0. Preliminaries

Course material:

The entire course material is available on the course web site:

http://www.staff.city.ac.uk/ fring/MathMeth/

This includes the lecture notes, exercise sheets, course work sheets, past papers and some relevant links.

Assessment:

There will be 1 coursework counting 15% a class test counting 5% and an exam in May counting 80% towards the final mark.

All marks will be reported on Moodle.

Books:

The notes should be self-contained but there are also useful books:

- Complex variables : introduction and applications / M. J. Ablowitz, A. S. Fokas (Cambridge : Cambridge University Press, 2003)
- Complex variables and their applications / Anthony D. Osborne (Harlow : Addison Wesley Longman, 1999)
- Fundamentals of complex analysis with applications to engineering and science / E.B. Saff, A.D. Snider; (Upper Saddle River, NJ : Prentice Hall, c2003)
- Applied complex analysis with partial differential equations / N. H. Asmar, Gregory C. Jones (Upper Saddle River, N.J. ; London : Prentice Hall, c2002)
- Complex analysis : an introduction to the theory of analytic functions of one complex variable / Lars V. Ahlfors (New York ; London : McGraw-Hill, 1979)

All books are available in the City library.

Structure of the course:

The course has three main sections:

I Complex analysis with an emphasis on conformal mappings.

II Application to boundary value problems.

III Transform methods and their applications to differential equations.

1. Complex Analysis

1.1. Complex Algebra

How to handle complex numbers?

Definition: *A complex number (variable) denoted by z is an ordered pair of real numbers (variables) $x, y \in \mathbb{R}$*

$$z = (x, y) \qquad \text{or} \qquad z = x + iy$$

with $i = \sqrt{-1}$.

$x = \mathrm{Re}\, z$ *is called the real part of z and*

$y = \mathrm{Im}\, z$ *is called the imaginary part of z.*

Ordered means that $(x, y) \neq (y, x)$.

1.1.1 Arithmetic operations, the field $\mathbb{C}$

How to compute with complex numbers? Take any two complex numbers

$$z = x + iy \qquad \text{and} \qquad w = u + iv$$

Addition:

$$z + w = (x + iy) + (u + iv) = (x + u) + i(y + v) \in \mathbb{C}$$

Multiplication:

$$z \cdot w = (x + iy) \cdot (u + iv) = (xu - yv) + i(yu + xv) \in \mathbb{C}$$

Division:

Assume

$$\frac{z}{w} = s + it \in \mathbb{C} \qquad w \neq 0 \tag{1}$$

Find s and t, if they exist. From (1) follows

$$z = x + iy = w \cdot (s + it) = (su - vt) + i(sv + ut).$$

Equate the real and imaginary parts

$$x = su - vt \qquad \text{and} \qquad y = sv + ut,$$

Solve for s and t

$$s = \frac{xu + yv}{u^2 + v^2} \qquad \text{and} \qquad t = \frac{yu - xv}{u^2 + v^2}. \tag{2}$$

Therefore

$$\frac{z}{w} = \frac{xu + yv}{u^2 + v^2} + i\left(\frac{yu - xv}{u^2 + v^2}\right) = \frac{z\bar{w}}{w\bar{w}} \qquad w \neq 0. \tag{3}$$

Since $x, y, u, v \in \mathbb{R} \Rightarrow s, t \in \mathbb{R} \Rightarrow \frac{z}{w} \in \mathbb{C}$

Therefore like $\mathbb{Q}$ and $\mathbb{R}$, the complex numbers also constitute a *field*, which is denoted by $\mathbb{C}$.

Definition: *A set of objects is said to be a <u>field</u> if the addition and multiplication is well defined and if for all z, w and s we have*

commutativity:	$z + w = w + z$	$z \cdot w = w \cdot z$
associativity:	$z + (w + s) = (z + w) + s$	$z \cdot (w \cdot s) = (z \cdot w) \cdot s$
distributivity:	$z \cdot (w + s) = z \cdot w + z \cdot s$	

a zero element exists
every non-zero element has an inverse with respect to $\cdot$ and $+$.

For $\mathbb{C}$:
$\cdot$ and $+$ are well defined
commutativity, associativity and distributivity are easily checked
the zero element is $(0, 0) = 0 + i0 = 0$
the identity element is $(1, 0) = 1 + i0 = 1$.

Note: $\mathbb{Z}$ is only a ring since the last requirement does not hold.
For instance: $7 \in \mathbb{Z}$ but the inverse element $1/7 \notin \mathbb{Z}$.)

1.1.2 Complex conjugation and absolute value

Definition: *The operation which sends* $z = x + iy$ *into* $\bar{z} = x - iy$ *is called complex conjugation. We say* $\bar{z}$ (or z^*) *is the conjugate of* z*.*

Clearly

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \text{and} \qquad \overline{z \cdot w} = \bar{z} \cdot \bar{w}\ .$$

The complex conjugation is an *involutory transformation*, that is

$$\bar{\bar{z}} = z.$$

Definition: *The modulus or absolute value of a complex number* $z = x + iy$ *is defined as*

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2} \geq 0$$

We have

$$|z \cdot w| = \sqrt{(z \cdot w)(\overline{z \cdot w})} = \sqrt{(z \cdot w)(\bar{z} \cdot \bar{w})} = \sqrt{(z \cdot \bar{z})(w \cdot \bar{w})} = |z| \cdot |w|\ ,$$

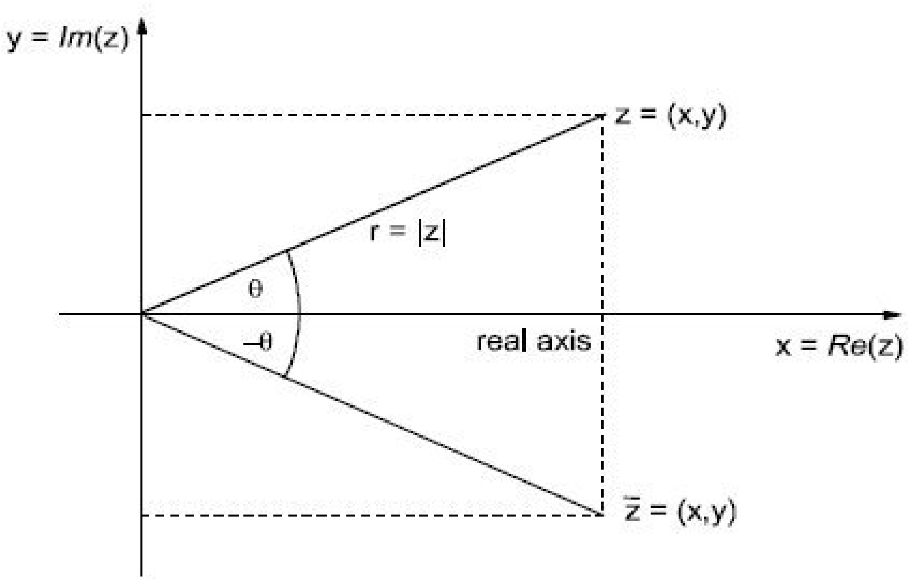
and the triangle inequalities

$$|z| - |w| \leq |z + w| \leq |z| + |w| \tag{4}$$

For the proof of (4) see sheet 1 task 2.

1.1.3. The Gauß-plane, polar form

We can represent a complex number in the *complex*: *(Gauß)-plane*:



From the figure

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

With Euler's formula we write z in *polar form* as

$$\boxed{z = r\cos\theta + ir\sin\theta = re^{i\theta}}\,.$$

Graphical interpretation:

- $|z|$ is the distance between the origin and the point (x, y).
- The angle θ is called the *argument* of z, i.e.

$$\boxed{r = |z|} \qquad \text{and} \qquad \boxed{\theta = \arg z = \arctan \frac{y}{x}}.$$

- The complex conjugation is a reflection about the real axis.

Note:
$\arg z$ is multi-valued as all $\theta_n = \theta + 2\pi n$ for $n \in \mathbb{Z}$ give the same z.
A unique so-called *principle value* is selected by convention.
For instance the choice $\theta = \theta_0 + 2\pi n$ with $-\pi < \theta_0 \leq \pi$ with a specific value for n, say $n = 0$ gives only one definite value.
We adopt here this convention.

Example 1: First convert every fraction of two rational numbers into the form $z = x + iy$. Then find the Gauß form

$$\frac{4}{1-i\sqrt{3}} = \frac{4\overline{(1-i\sqrt{3})}}{(1-i\sqrt{3})\overline{(1-i\sqrt{3})}} = 1+i\sqrt{3} = 2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}) = 2e^{i\pi/3}.$$

Example 2: We can write z^n for $r = 1$ in two alternative ways

$$z^n = e^{in\theta} = \cos n\theta + i\sin n\theta = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n.$$

This is the *de Moivre formula*.

Using complex numbers this identity was trivial to prove, whereas more effort is needed in a purely trigonometric setting.

1.1.4. The n-roots of z

To compute the n-th root of a complex number $z_0 = z^{1/n}$ we solve

$$z_0^n = z \tag{5}$$

for z_0. With $z_0 = r_0 \exp(i\theta_0)$ and $z = r \exp(i\theta)$ equation (5) reads

$$r_0^n e^{in\theta_0} = re^{i\theta}$$

therefore

$$r_0 = \sqrt[n]{r} \quad \text{and} \quad \theta_0 = \frac{\theta}{n} + \frac{2k\pi}{n} \quad \text{for } k \in \mathbb{Z}.$$

Therefore the n distinct solutions of (5) are

$$\boxed{z_0^{(k)} = \sqrt[n]{r}\left[\cos\left(\tfrac{\theta+2k\pi}{n}\right) + i\sin\left(\tfrac{\theta+2k\pi}{n}\right)\right] = \sqrt[n]{r}\, e^{\frac{\theta+2k\pi}{n}i}} \quad \text{for } k = 0, 1, \ldots, n-1.$$

Special case: n-th root of unity for $z = 1$, i.e. $r = 1$ and $\theta = 0$

$$z_0^{(k)} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) = e^{\frac{2k\pi}{n}i} \quad \text{for } k = 0, 1, \ldots, n-1.$$

One usually denotes $z_0^{(1)} =: \omega$, such that we simply have $z_0^{(2)} = \omega^2$, $z_0^{(3)} = \omega^3$, etc.

1.2 Analytic functions
1.2.1 Functions of a complex variable

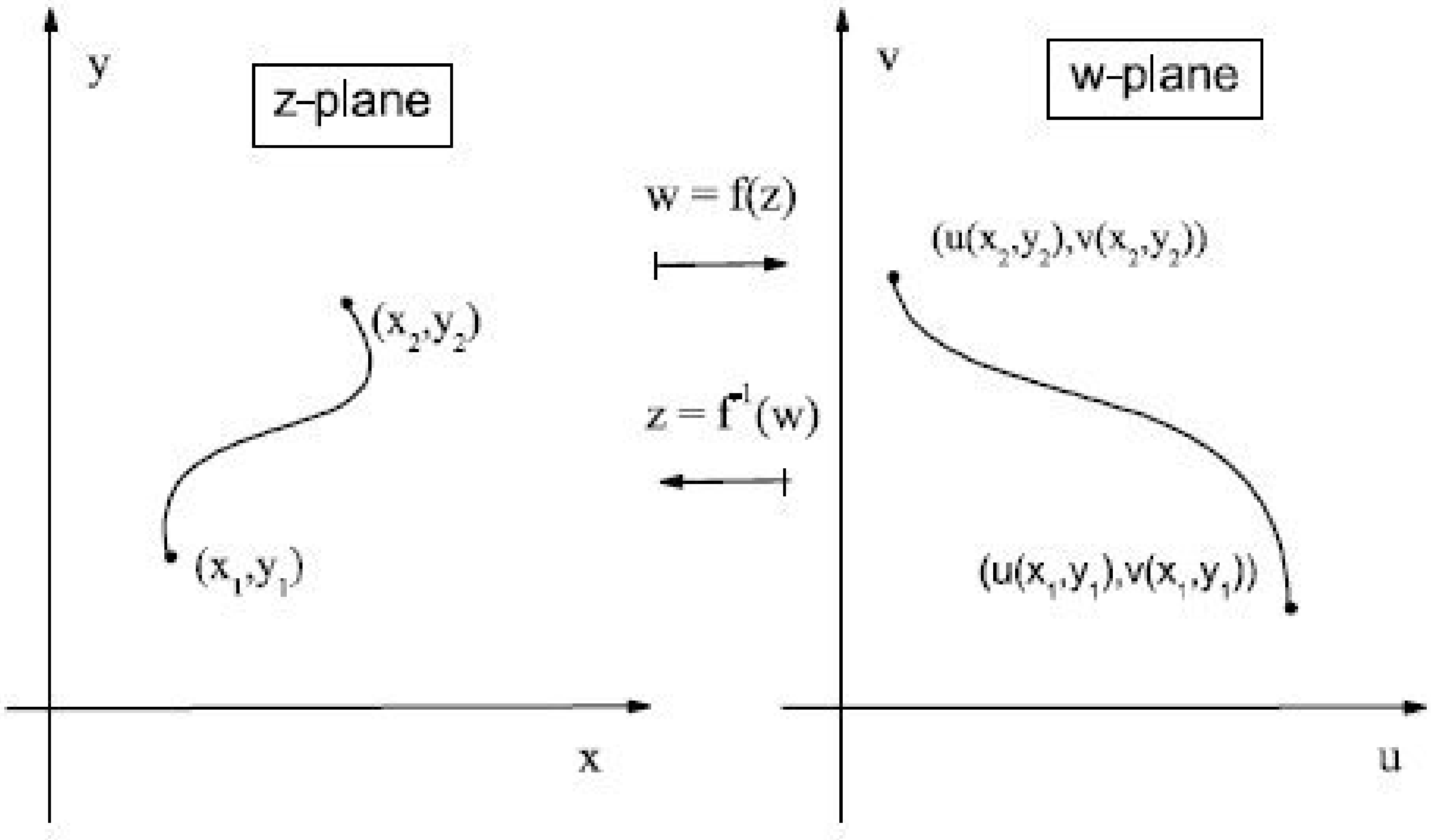
Definition: *The map*

$$\begin{aligned} f : z \mapsto w &= f(z) \\ &= u(x,y) + iv(x,y) \end{aligned} \tag{6}$$

which assigns to each complex number $z = x + iy \in D \subset \mathbb{C}$ exactly one other complex number is called a function of a complex variable.

- *D is called the domain of definition.*
- *The totality of all possible values $f(z)$ for all $z \in D$ is called the range.*
- *The map $f^{-1} : w \mapsto z = f^{-1}(w)$ is called the inverse of f.*
- *A point $z_0 \in D$, which is mapped by f onto itself is called a fixed point, i.e. $w = f(z_0) = z_0$.*

We have the following picture in mind:



Example 1: The domain of the complex valued function

$$w = f(z) = \frac{1}{z^2+1}$$

is $D = \mathbb{C}\backslash\{\pm i\}$.
Example 2: The real and imaginary part of the complex valued function is computed as

$$\begin{aligned} w &= f(z) = z^2 = (x+iy)^2 \\ &\Rightarrow u(x,y) = x^2 - y^2, v(x,y) = 2xy. \end{aligned}$$

Example 3: The fixed point of

$$w = f(z) = \frac{6z-9}{z}$$

is $z_0 = 3$. This follows from $f(z_0) = z_0 \Leftrightarrow z_0^2 - 6z_0 + 9 = 0$.
Example 4: The inverse function of $w = f(z) = 2z - 4$ is

$$z = f^{-1}(w) = \frac{w}{2} + 2.$$

This follows from exchanging $z \leftrightarrow w$, that is solving $z = 2w - 4$. We may also verify that $f(f^{-1}(z)) = z$ and $f^{-1}(f(z)) = z$.

Example 5: The inverse function of $w = f(z) = \exp z = r\exp(i\theta)$ is

$$z = \ln r + i\theta + 2\pi i n \qquad \text{with } n \in \mathbb{Z}. \tag{7}$$

We note in example 5 that there is not a one-to-one correspondence between values in the domain and the range. Such functions have a special name:

Definition: *A multivalued function acquires more than one value in its range for at least one value in its domain.*

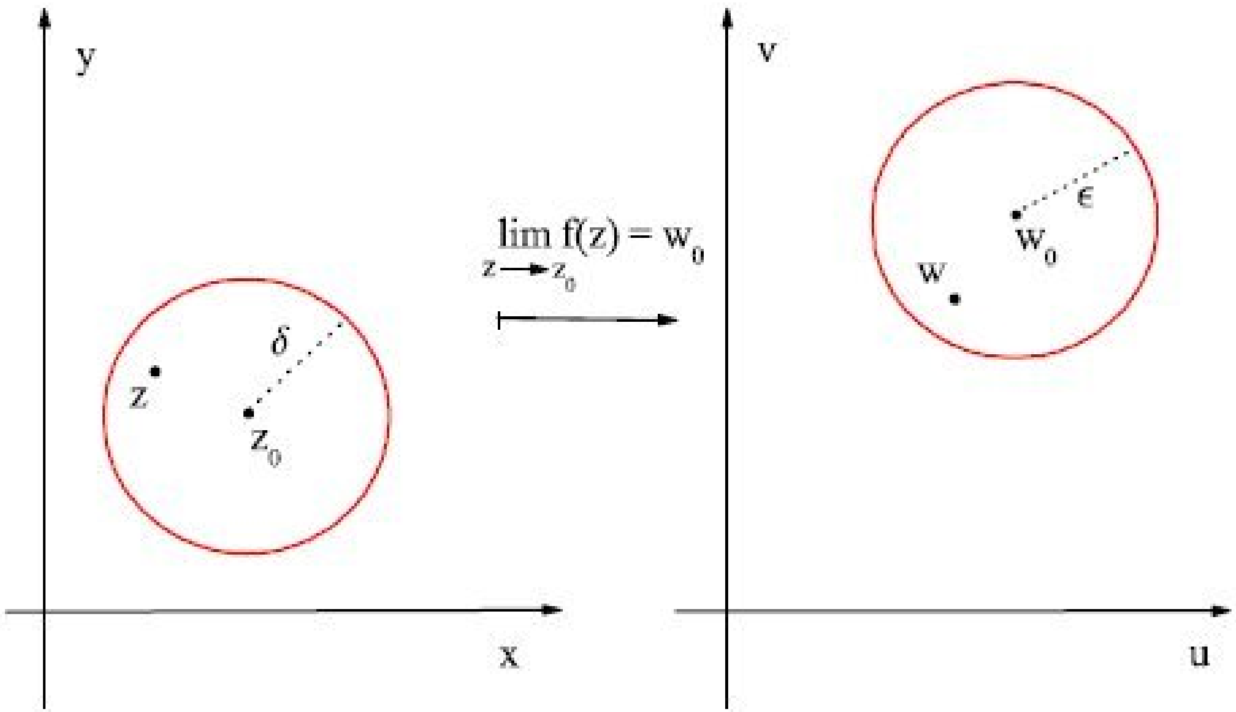
We will see later how to cure this.

1.2.2 Limits, Continuity and Complex derivatives

Definition: *The function $f(z)$ is said to possess the* <u>*limit*</u> *w_0 as z tends to z_0*

$$\lim_{z \to z_0} f(z) = w_0 \tag{8}$$

iff for every $\epsilon > 0$ there exists a $\delta > 0$, such that $|f(z) - w_0| < \epsilon$ for all values of z for which $|z - z_0| < \delta, z \neq z_0$.



Example: Use the previous definition to argue that

$$\lim_{z \to 3} \frac{z^2 - 9}{z - 3} = 6.$$

Solution :

- The domain for $f(z) = (z^2 - 9)/(z - 3)$ is $D = \mathbb{C} \backslash \{3\}$, which means that $f(z = 3)$ is not defined.
- On D we have $f(z) = (z + 3)$, such that

$$|f(z) - 6)| = |z + 3 - 6| = |z - 3| \qquad \text{for} \;\; z \neq 3.$$

- This means for every $\epsilon > 0$ for which $|f(z) - 6)| < \epsilon$ there exists a $\delta = \epsilon > 0$ for which $|z - 3| < \delta$.
- Therefore we have $\lim_{z \to z_0 = 3} f(z) = 6$.

More practical:

Theorem 1: *Introducing the following quantities*

$$f(z) = u(x,y) + iv(x,y), \quad z = x + iy, \quad z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0$$

the limit

$$\lim_{z \to z_0} f(z) = w_0$$

exists iff

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0 \qquad \text{and} \qquad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0.$$

Proof : We omit this here.

Definition: *The function $f(z)$ is said to be continuous at the point z_0 iff* $\lim_{z \to z_0} f(z) = f(z_0)$.

Definition: *Let f be a function defined on some domain $D \subset \mathbb{C}$, with $z_0 \in D$. Then f is said to be (complex) differentiable if there exists a continuous function f':$D \to \mathbb{C}$ for all $z \in D$*

$$\boxed{f'(z_0) = \left.\frac{df}{dz}\right|_{z=z_0} = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}},$$

with $h = \Delta z = z - z_0$. f' is called the derivative of f.

Note that unlike as for real valued functions we have now various options to take the limit $h \to 0$.

1.2.3 Analyticity and the Cauchy-Riemann equations

Suppose that the limit $f'(z_0)$ exists. Then we can write

$$f'(z_0) = \lim_{\substack{h_x \to 0 \\ h_y \to 0}} \frac{u(x_0 + h_x, y_0 + h_y) - u(x_0, y_0) + iv(x_0 + h_x, y_0 + h_y) - iv(x_0, y_0)}{h_x + ih_y},$$

with $f(z) = u(x, y) + iv(x, y)$ and $h = h_x + ih_y$.
Now we have two options to take the limit, either in the order $h_y \to 0$ and then

$$\begin{aligned} f'(z_0) &= \lim_{h_x \to 0} \frac{u(x_0 + h_x, y_0) - u(x_0, y_0) + iv(x_0 + h_x, y_0) - iv(x_0, y_0)}{h_x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned} \tag{9}$$

or to take first the limit $h_x \to 0$ and then

$$
\begin{aligned}
f'(z_0) &= \lim_{h_y \to 0} \frac{u(x_0, y_0 + h_y) - u(x_0, y_0) + iv(x_0, y_0 + h_y) - iv(x_0, y_0)}{ih_y} \\
&= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \qquad (10)
\end{aligned}
$$

We used here Theorem 1, which allows us to split the limit for u and v. Comparing (9) and (10) we find the
Cauchy-Riemann equations (conditions)

$$
\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \text{and} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}. \qquad (11)
$$

From the above argument it is clear that the Cauchy-Riemann condition is a *necessary* condition for the derivative $f'(z_0)$ to exist. The following theorem provides also a *sufficient* condition.

Theorem 2: *Suppose that for a function $f(z) = u(x, y) + iv(x, y)$ all four partial derivatives of u and v are continuous at the point z_0 and in addition satisfy the Cauchy-Riemann condition, then the derivative $f'(z_0)$ exists.*

Proof : We omit this here.

Example: We consider once more the function

$$f(z) = z^2 = u(x, y) + iv(x, y) = x^2 - y^2 + i2xy$$

We verify the Cauchy-Riemann condition

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x},$$

and therefore

$$f'(z) = 2x + i2y = 2z.$$

Definition: *A function f of a complex variable z is said to be analytic in the domain $D \subset \mathbb{C}$ if its derivative exists for all $z \in D$.*
A function is said to be analytic in the point z_0 if there exists a neighbourhood around z_0 in which f is analytic.
When $D = \mathbb{C}$ the function is called an entire function.

Theorem 3:
The derivative of an analytic function is also an analytic function.
Proof : We omit this here.

1.2.4. Harmonic functions and the Laplace equation

Definition: *A function u which satisfies the Laplace equation $\Delta u = 0$ is said to be a harmonic function.*

A function v is said to be the conjugate harmonic function of u, if they are both harmonic functions and satisfy the Cauchy-Riemann equations.

Corollary 1: *The real and imaginary parts of an analytic function are harmonic functions. Conversely, if the two functions $u(x, y)$ and $v(x, y)$ are harmonic functions then $f(z) = u(x, y) + iv(x, y)$ is an analytic function.*

Proof :

- We differentiate the Cauchy-Riemann equations with respect to x and y

$$\partial_x^2 u = \partial_x \partial_y v \qquad \text{and} \qquad \partial_y^2 u = -\partial_x \partial_y v,$$

  respectively.
- Theorem 3 guarantees that the second derivatives exist.
- Adding these two equation gives

$$\Delta u = (\partial_x^2 + \partial_y^2) u = 0.$$

- Similarly differentiating the Cauchy-Riemann equations with respect to y and x instead gives $\Delta v = 0$.
- The converse is shown by integration.

Example 1: Once more we consider the function

$$f(z) = z^2 = u(x,y) + iv(x,y) = x^2 - y^2 + i2xy$$

Clearly u and v are harmonic functions

$$\begin{aligned} \Delta u &= \partial_x^2 u + \partial_y^2 u = \partial_x(2x) - \partial_y(2y) = 0, \\ \Delta v &= \partial_x^2 v + \partial_y^2 v = \partial_x(2y) - \partial_y(2x) = 0. \end{aligned}$$

Example 2: Consider

$$f(z) = z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3) = u(x,y) + iv(x,y)$$

It is easy to see that u and v are harmonic functions

$$\begin{aligned} \Delta u &= \partial_x^2 u + \partial_y^2 u = \partial_x(3x^2 - 3y^2) + \partial_y(-6xy) = 6x - 6x = 0, \\ \Delta v &= \partial_x^2 v + \partial_y^2 v = \partial_x(6xy) + \partial_y(6x - 3y^2) = 6y - 6y = 0. \end{aligned}$$

Example 3: Given a harmonic function one can use the Cauchy-Riemann equations to compute its conjugate harmonic function and thereafter construct an analytic function.
For instance taking $u(x,y) = \cosh x \cos y$ it follows

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \cos y \sinh x = \frac{\partial v}{\partial y} \\
\Rightarrow v &= \sinh x \int \cos y dy = \sinh x \sin y + \sinh x g(x), \\
\frac{\partial u}{\partial y} &= -\sin y \cosh x = -\frac{\partial v}{\partial x} \Rightarrow \\
v &= \sin y \int \cosh x dx = \sinh x \sin y + \sinh y h(y),
\end{aligned}$$

such that $g(x) = h(x) = 0$. The conjugate harmonic function is therefore $v(x,y) = \sinh x \sin y$.
Hence $f(x,y) = \cosh x \cos y + i \sinh x \sin y$ is an analytic function by corollary 1.