Mathematical Methods II

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ABSTRACT: This is an introduction into frequently used mathematical methods focusing on complex analysis. The course consists of three main parts. In the first part we study basic topics in complex analysis with an emphasis on conformal mappings. In the second part we apply these techniques to solve boundary value problems of Dirichlet and Neumann type. In the third part we discuss transform methods, in particular Fourier and Lagrange transformations, and their applications to differential equations.

1. Complex Analysis

"*The imaginary number takes mathematics to another dimension. It was discovered in sixteenth century Italy at a time when being a mathematician was akin to being a modern day rock star, when there was 'nuff respect' to be had from solving a particularly 'wicked' equation. And the wicked equation of the day went like this: 'If the square root of +1 is both +1 and -1, then what is the square root of -1?' Previously, mathematicians had rolled their eyes skyward and prayed for divine intervention. But where others failed, the creative Italian Rafaello Bombelli triumphed with his invention of the imaginary number. The imaginary number is the square root of -1 and is known as 'i'.*" Simon Singh

1.1 Complex Algebra

First we have to learn how to handle complex numbers. Recall

Definition: *A complex number (variable) denoted by z is an ordered pair of real numbers (variables) $x, y \in \mathbb{R}$*

$$z = (x, y) \qquad \text{or} \qquad z = x + iy \tag{1.1}$$

with $i = \sqrt{-1}$. Here $x = \operatorname{Re} z$ is called the real part of z and $y = \operatorname{Im} z$ is called the imaginary part of z.

Ordered means that $(x, y) \neq (y, x)$. One could just provide some rules for the expressions in the form (x, y), but it will be more convenient for us to use the second equation in (1.1), which is completely equivalent.

1.1.1 Arithmetic operations, the field $\mathbb{C}$

We will now see how to compute with complex numbers. Let us take for this purpose two complex numbers of the form

$$z := x + iy \qquad \text{and} \qquad w := u + iv \qquad \text{with } x, y, u, v \in \mathbb{R}, \tag{1.2}$$

and investigate how combine subject to various arithmetic operations. The *addition* of these numbers yields again a complex number of the form (1.1) as we can see easily

$$z + w = (x + iy) + (u + iv) = (x + u) + i(y + v). \tag{1.3}$$

Obviously the expression on the right hand side is again of the form (1.1), since $(y + v)$, $(x + u) \in \mathbb{R}$. Likewise the *multiplication* of two complex numbers gives again a complex number

$$z \cdot w = (x + iy) \cdot (u + iv) = (xu - yv) + i(yu + xv), \tag{1.4}$$

since $(xu-yv)$, $(yu+xv) \in \mathbb{R}$. That is also holds for the *division* is slightly less obvious. Let us now verify that this is in fact also a meaningful composition of two complex numbers. Suppose that this operation is of the form

$$\frac{z}{w} = s + it \qquad w \neq 0; s, t \in \mathbb{R}, \tag{1.5}$$

and let us determine s and t, if they exist. From (1.5) follows

$$x + iy = w \cdot (s + it) = (su - vt) + i(sv + ut). \tag{1.6}$$

Equating in (1.6) the real and imaginary parts on both sides of the equation gives the two coupled linear equations

$$x = su - vt \qquad \text{and} \qquad y = sv + ut, \tag{1.7}$$

which we can solve uniquely for s and t. We easily find

$$s = \frac{xu + yv}{u^2 + v^2} \qquad \text{and} \qquad t = \frac{yu - xv}{u^2 + v^2}. \tag{1.8}$$

Clearly s, $t \in \mathbb{R}$, since x, y, u, $v \in \mathbb{R}$ and the addition, multiplication and division of real numbers produces real numbers. Therefore the division of two complex numbers results indeed into another complex number

$$\frac{z}{w} = \frac{xu + yv}{u^2 + v^2} + i\left(\frac{yu - xv}{u^2 + v^2}\right) = \left(\frac{z\bar{w}}{w\bar{w}}\right) \qquad w \neq 0. \tag{1.9}$$

We have indicated here that for practical purposes we simply have to multiply the nominator and denominator by the conjugate of the denominator $\bar{w}$ (see section 1.1.2 for definition) in order to obtain a complex number in the form (1.1).

This means just like the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$, the complex numbers also constitute what is referred to as a *field*, denoted usually by $\mathbb{C}$. For a set of objects to be a field we require well defined addition and multiplication. In addition, we have to verify for all elements z, w and s in the field that the following properties hold

$$\begin{array}{rccr} \text{commutativity:} & z+w=w+z & z\cdot w=w\cdot z, & \\ \text{associativity:} & z+(w+s)=(z+w)+s & z\cdot(w\cdot s)=(z\cdot w)\cdot s, & (1.10) \\ \text{distributivity:} & z\cdot(w+s)=z\cdot w+z\cdot s, & & \end{array}$$

that a zero element exists and we also have to guarantee that every non-zero element has an inverse with respect to multiplication and addition. For the field $\mathbb{C}$ we have seen that addition and multiplication are well defined. The properties (1.10) are easily verified. The zero element is $(0,0) = 0 + i0 = 0$, the identity element is $(1,0) = 1 + i0 = 1$ and distributivity is checked by direct computation.

(Recall that the integers $\mathbb{Z}$ only constitute a ring, but not a field, as the last requirement does not hold. For instance, taking $7 \in \mathbb{Z}$ we find that the inverse element $1/7 \notin \mathbb{Z}$.)

1.1.2 Complex conjugation and absolute value

Definition: *The operation which sends $z = x + iy$ into $\bar{z} := x - iy$ is called complex conjugation. We say $\bar{z}$ is the conjugate of z. (Sometimes also the notation z^* is used for $\bar{z}$.)*

Clearly we have the following rules

$$\overline{z+w} = \bar{z} + \bar{w} \qquad \text{and} \qquad \overline{z \cdot w} = \bar{z} \cdot \bar{w}\ , \tag{1.11}$$

which follow by direct computation. Furthermore, we may convince ourselves that the complex conjugation is an *involutory transformation*, that is applying it twice yields the original argument

$$\bar{\bar{z}} = z. \tag{1.12}$$

Definition: *The modulus or absolute value of a complex number $z = x + iy$ is defined as*

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2} \geq 0 \tag{1.13}$$

We have

$$|z \cdot w| = \sqrt{(z \cdot w)(\overline{z \cdot w})} = \sqrt{(z \cdot w)(\bar{z} \cdot \bar{w})} = \sqrt{(z \cdot \bar{z})(w \cdot \bar{w})} = |z| \cdot |w|\ , \tag{1.14}$$

and the triangle inequalities

$$|z| - |w| \leq |z + w| \leq |z| + |w| \tag{1.15}$$

Proof : To prove (1.15) we start with

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w}) = z\bar{z} + w\bar{z} + z\bar{w} + w\bar{w} = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}). \tag{1.16}$$

Therefore we deduce

$$|z+w|^2 - (|z| + |w|)^2 = -2\,|z|\,|w| + 2\,\mathrm{Re}(z\bar{w}), \tag{1.17}$$

Since $\mathrm{Re}(z\bar{w}) \leq |z\bar{w}| = |z|\,|w|$ it follows that

$$|z+w|^2 - (|z|+|w|)^2 \leq 0, \tag{1.18}$$

which when taking the square root yields the second inequality in (1.15). The first inequality is trivial. $\square$

Alternatively we may think of a complex number as a two-dimensional vector $\vec{z} = (x, y)$ then these identities just become the usual relation for vectors in more generality. (recall $|\vec{z}| = \sqrt{\vec{z} \cdot \vec{z}} = x^2 + y^2$, see e.g. MA1607 Geometry & Vectors).

1.1.3 The Gauß-plane, polar form

For various reasons which we will discuss in more detail below it is convenient to represent complex numbers in the *complex*: *(Gauß[1])-plane*: From figure 2 we read off directly that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \tag{1.19}$$

Figure 1: Gauß on the 10 D-Mark note

which suggests, using Euler's [2] formula, that we can write z in the so-called *polar form* (or Gauß-form) as

$$\boxed{z = r\cos\theta + ir\sin\theta = re^{i\theta}\ .} \tag{1.20}$$

The angle θ is called the *argument* of z. We measure it in the so-called positive mathematical sense, that is anti-clockwise, against the horizontal as indicated in figure 2. Note that $\arg z$ is multivalued as all $\theta_n = \theta + 2\pi n$ for $n \in \mathbb{Z}$ give the same z. A unique so-called *principle value* is selected by convention. For instance, for a specific value of n the choice $\theta = \theta_0 + 2\pi n$ with $-\pi < \theta_0 \leq \pi$ gives only one definite value. We adopt here this convention. The distance r between the origin and the point (x, y) in the Gauß-plane is then given by the modulus of z. Thus, for any complex number given in the standard form $z = x + iy$ we may compute r and θ from the expressions

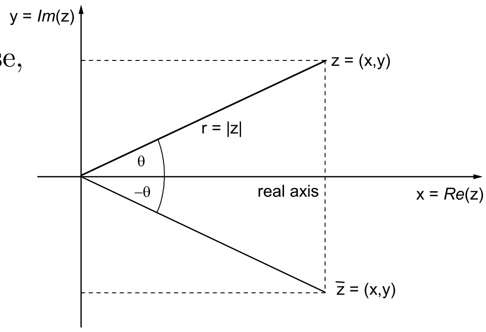


Figure 2: Complex (Gauß)-plane

$$\boxed{r = |z| = \sqrt{x^2+y^2}} \qquad \text{and} \qquad \boxed{\theta = \arg z = \arctan\tfrac{y}{x}}, \tag{1.21}$$

which follow immediately from (1.19).

The complex conjugation $\bar{z} = re^{-i\theta}$ of $z = re^{i\theta}$corresponds therefore to a reflection about the real axis as indicated in figure 2.

Some useful simple numbers in polar and normal form are: $1 = e^{i0} = e^{2\pi i}$, $-1 = e^{i\pi} = e^{-i\pi}$, $i = e^{i\pi/2}$, $-i = e^{-i\pi/2}$, $(1+i\sqrt{3})/2 = e^{i\pi/3}$, $(1+i)/\sqrt{2} = e^{i\pi/4}$ and $(\sqrt{3}+i)/2 = e^{i\pi/6}$. These numbers are worthwhile to remember as they emerge again and again.

Let us consider a few examples in order to see the working of these formulae:

Example 1: Using equation (1.9) we can convert every fraction of two rational numbers into the form $z = x + iy$. Subsequently we use (1.20) to obtain the Gauß form

$$\frac{4}{1-i\sqrt{3}} = \frac{4\overline{(1-i\sqrt{3})}}{(1-i\sqrt{3})\overline{(1-i\sqrt{3})}} = 1 + i\sqrt{3} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 2e^{i\pi/3}. \tag{1.22}$$

Example 2: In view of (1.20) we can write z^n for $r = 1$ in two alternative ways

$$z^n = e^{in\theta} = \underline{\cos n\theta + i\sin n\theta} = \left(e^{i\theta}\right)^n = \underline{(\cos\theta + i\sin\theta)^n}, \tag{1.23}$$

which is the <u>*de Moivre formula*</u>. Using complex numbers this identity is trivial to prove, whereas more effort is needed in a purely trigonometric setting.

1.1.4 The n-roots of z

Computing the n-th root of a complex number $z^{1/n} =: z_0$ amounts to solving the equation

$$z_0^n = z \tag{1.24}$$

for z_0. Thus when expressing the complex numbers in (1.24) in polar form $z_0 = r_0\exp(i\theta_0)$ and $z = r\exp(i\theta)$, the equation reads

$$r_0^n e^{in\theta_0} = re^{i\theta}, \tag{1.25}$$

and therefore we have

$$r_0 = \sqrt[n]{r} \quad \text{and} \quad \theta_0 = \frac{\theta}{n} + \frac{2k\pi}{n} \quad \text{for } k \in \mathbb{Z}. \tag{1.26}$$

Thus the n distinct solutions of (1.24) are

$$\boxed{z_0^{(k)} = \sqrt[n]{r}\left[\cos\left(\tfrac{\theta+2k\pi}{n}\right) + i\sin\left(\tfrac{\theta+2k\pi}{n}\right)\right] = \sqrt[n]{r}e^{\frac{\theta+2k\pi}{n}i}} \quad \text{for } k = 0, 1, \ldots, n-1. \tag{1.27}$$

A special case of this are the n-th root of unity for $z = 1$, i.e. $r = 1$ and $\theta = 0$

$$z_0^{(k)} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) = e^{\frac{2k\pi}{n}i} \quad \text{for } k = 0, 1, \ldots, n-1. \tag{1.28}$$

One usually denotes $z_0^{(1)} = e^{\frac{2\pi i}{n}} =: \omega$, such that we simply have $z_0^{(2)} = \omega^2$, $z_0^{(3)} = \omega^3$, etc.

Note that unlike as for real numbers we therefore have that $\sqrt[n]{1} \neq 1$. For instance the following identities hold: $1^{1/3} = (e^{2\pi i})^{1/3} = e^{2\pi i/3} = (e^{i\pi})^{2/3} = (-1)^{2/3}$.

1.2 Analytic functions

Having established how to manipulate complex numbers, let us see next how to deal with functions of these numbers and study some of their basic properties.

1.2.1 Functions of a complex variable

Definition: *The map*

$$f : z \mapsto w = f(z) \qquad (1.29)$$
$$= u(x, y) + iv(x, y)$$

which assigns to each complex number $z = x + iy \in D \subset \mathbb{C}$ exactly one other complex number is called a function of a complex variable.
D is called the domain of definition.
The totality of all possible values $f(z)$ for all values $z \in D$ is called the range.

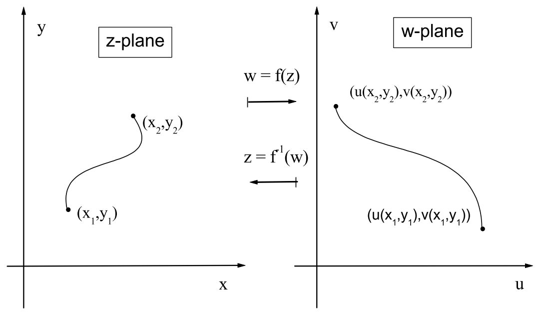


Figure 3: The function $f : z \mapsto w = f(z)$.

The map $f^{-1} : w \mapsto z = f^{-1}(w)$ is called the inverse of f.
A point $z_0 \in D$, which is mapped by f onto itself is called a fixed point, i.e. $w = f(z_0) = z_0$.

Example 1: The domain of the complex valued function

$$w = f(z) = \frac{1}{z^2 + 1} \qquad (1.30)$$

is $D = \mathbb{C}\backslash\{\pm i\}$.

Example 2: We compute the real and imaginary part of the complex valued function

$$w = f(z) = z^2 = (x + iy)^2 \qquad (1.31)$$
$$\Rightarrow u(x, y) = x^2 - y^2, v(x, y) = 2xy.$$

Example 3: The fixed point of

$$w = f(z) = \frac{6z - 9}{z} \qquad (1.32)$$

is $z_0 = 3$, which follows from $f(z_0) = z_0 \Leftrightarrow z_0^2 - 6z_0 + 9 = 0$.

Example 4: The inverse function of $w = f(z) = 2z - 4$ is

$$z = f^{-1}(w) = \frac{w}{2} + 2, \qquad (1.33)$$

which follows simply from exchanging $z \leftrightarrow w$, that is solving $z = 2w - 4$. We may also verify that $f(f^{-1}(z)) = z$ and $f^{-1}(f(z)) = z$.

Example 5: The inverse function of $w = f(z) = \exp z = r \exp(i\theta)$ is

$$z = \ln r + i\theta + 2\pi in \qquad \text{with } n \in \mathbb{Z}. \qquad (1.34)$$

We note here that there is not a one-to-one correspondence between values in the domain and the range. Such functions have a special name:

Definition: *A multivalued function acquires more than one value in its range for at least one value in its domain.*

We shall see later how to remedy such kind of behaviour and convert such functions into proper single valued functions as defined in (1.29).

1.2.2 Limits, Continuity and Complex derivatives

Definition: *The function $f(z)$ is said to possess the limit w_0 as z tends to z_0*

$$\lim_{z \to z_0} f(z) = w_0 \tag{1.35}$$

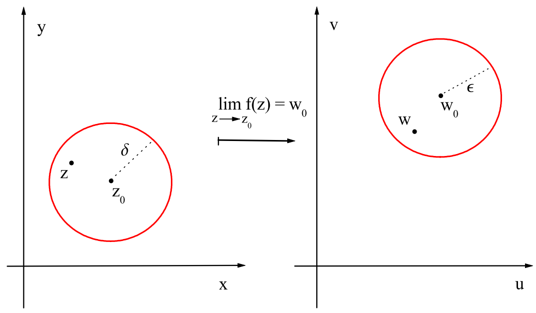


Figure 4: The limit of $f(z)$ as $z \to z_0$.

iff for every $\epsilon > 0$ there exists a $\delta > 0$, such that $|f(z) - w_0| < \epsilon$ for all values of z for which $|z - z_0| < \delta, z \neq z_0$.

Example: Use the previous definition to argue that

$$\lim_{z \to 3} \frac{z^2 - 9}{z - 3} = 6. \tag{1.36}$$

Solution : The domain for $f(z) = (z^2 - 9)/(z - 3)$ is $D = \mathbb{C}\backslash\{3\}$, which means that $f(z = 3)$ is not defined. On D we have $f(z) = (z + 3)$, such that

$$|f(z) - 6)| = |z + 3 - 6| = |z - 3| \qquad \text{for} \quad z \neq 3. \tag{1.37}$$

This means for every $\epsilon > 0$ for which $|f(z) - 6)| < \epsilon$ there exists a $\delta = \epsilon > 0$ for which $|z - 3| < \delta$. Therefore we have $\lim_{z \to z_0 = 3} f(z) = 6$.

The following theorem provides an alternative and more practical way to compute the limit defined in (1.35).

Theorem 1: *Introducing the following quantities*

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy, \quad z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0 \tag{1.38}$$

the limit

$$\lim_{z \to z_0} f(z) = w_0 \tag{1.39}$$

exists iff

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0 \qquad \text{and} \qquad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0. \tag{1.40}$$

Proof : We omit this here.

Definition: *The function $f(z)$ is said to be continuous at the point z_0 iff* $\lim_{z \to z_0} f(z) = f(z_0)$.

Definition: *Let f be a function defined on some domain $D \subset \mathbb{C}$, with $z_0 \in D$. Then f is said to be (complex) differentiable if there exists a continuous function f':$D \to \mathbb{C}$ for all $z \in D$*

$$\boxed{f'(z_0) = \left.\frac{df}{dz}\right|_{z = z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}}, \tag{1.41}$$

with $h = \Delta z = z - z_0$*.* f' *is called the* *derivative* *of* f*.*

Note that unlike as for real valued functions we have now various options to take the limit $h \to 0$, that means we have various pathes in the z-plane at our disposal. As a consequence of this ambiguity we obtain some non-trivial equations.

1.2.3 Analyticity and the Cauchy-Riemann equations

Suppose that the limit $f'(z_0)$ exists. Then we can write

$$f'(z_0) = \lim_{\substack{h_x \to 0 \\ h_y \to 0}} \frac{u(x_0 + h_x, y_0 + h_y) - u(x_0, y_0) + iv(x_0 + h_x, y_0 + h_y) - iv(x_0, y_0)}{h_x + ih_y}, \qquad (1.42)$$

where we expand in (1.41) $f(z) = u(x,y) + iv(x,y)$ and used $h = h_x + ih_y$. Now we have two options to take the limit, either in the order $h_y \to 0$ and then

$$f'(z_0) = \lim_{h_x \to 0} \frac{u(x_0 + h_x, y_0) - u(x_0, y_0) + iv(x_0 + h_x, y_0) - iv(x_0, y_0)}{h_x} \qquad (1.43)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (1.44)$$

or to take first the limit $h_x \to 0$ and then

$$f'(z_0) = \lim_{h_y \to 0} \frac{u(x_0, y_0 + h_y) - u(x_0, y_0) + iv(x_0, y_0 + h_y) - iv(x_0, y_0)}{ih_y} \qquad (1.45)$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \qquad (1.46)$$

We used here Theorem 1, which allows us to split the limit for u and v. Comparing (1.44) and (1.46) we find the *Cauchy-Riemann equations (conditions)* [3,4]

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \text{and} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}. \qquad (1.47)$$

From the above argument it is clear that the Cauchy-Riemann condition is a *necessary* condition for the derivative $f'(z_0)$ to exist. The following theorem provides also a *sufficient* condition.

Theorem 2: *Suppose that for a function* $f(z) = u(x,y) + iv(x,y)$ *all four partial derivatives of* u *and* v *are continuous at the point* z_0 *and in addition satisfy the Cauchy-Riemann condition, then the derivative* $f'(z_0)$ *exists.*

Proof : We omit this here.

Example: Once more we consider the function in (1.31)

$$f(z) = z^2 = u(x,y) + iv(x,y) = x^2 - y^2 + i2xy \qquad (1.48)$$

We verify the Cauchy-Riemann condition

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}, \qquad (1.49)$$

and therefore

$$f'(z) = 2x + i2y = 2z. \qquad (1.50)$$

Definition: *A function f of a complex variable z is said to be analytic in the domain $D \subset \mathbb{C}$ if its derivative exists for all $z \in D$. A function is said to be analytic in the point z_0 if there exists a neighbourhood around z_0 in which f is analytic. When $D = \mathbb{C}$ the function is called an entire function.*
Theorem 3: *The derivative of an analytic function is also an analytic function.*
Proof : We omit this here.

1.2.4 Harmonic functions and the Laplace equation

Definition: *A function u which satisfies the Laplace [5] equation $\Delta u = 0$ is said to be a harmonic function. A function v is said to be the conjugate harmonic function of u, if they are both harmonic functions and satisfy the Cauchy-Riemann equations (1.47).*
Corollary 1: *The real and imaginary parts of an analytic function are harmonic functions. Conversely, if the two functions $u(x,y)$ and $v(x,y)$ are harmonic functions then $f(z) = u(x,y) + iv(x,y)$ is an analytic function.*

Proof : We differentiate the Cauchy-Riemann equations with respect to x and y

$$\partial_x^2 u = \partial_x \partial_y v \qquad \text{and} \qquad \partial_y^2 u = -\partial_x \partial_y v, \tag{1.51}$$

respectively. Theorem 3 guarantees that the second derivatives exist. Adding these two equation yields

$$\Delta u = (\partial_x^2 + \partial_y^2) u = 0. \tag{1.52}$$

Similarly differentiating the Cauchy-Riemann equations with respect to y and x instead gives $\Delta v = 0$. The converse is shown by integration.

Example 1: Once more we consider the function in (1.31)

$$f(z) = z^2 = u(x,y) + iv(x,y) = x^2 - y^2 + i2xy \tag{1.53}$$

Clearly u and v are harmonic functions

$$\Delta u = \partial_x^2 u + \partial_y^2 u = \partial_x(2x) - \partial_y(2y) = 0, \tag{1.54}$$
$$\Delta v = \partial_x^2 v + \partial_y^2 v = \partial_x(2y) - \partial_y(2x) = 0. \tag{1.55}$$

Example 2: Consider

$$f(z) = z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2 y - y^3) = u(x,y) + iv(x,y) \tag{1.56}$$

It is easy to see that u and v are harmonic functions

$$\Delta u = \partial_x^2 u + \partial_y^2 u = \partial_x(3x^2 - 3y^2) + \partial_y(-6xy) = 6x - 6x = 0, \tag{1.57}$$
$$\Delta v = \partial_x^2 v + \partial_y^2 v = \partial_x(6xy) + \partial_y(6x - 3y^2) = 6y - 6y = 0. \tag{1.58}$$

Example 3: Given a harmonic function one can use the Cauchy-Riemann equations to compute its conjugate harmonic function and thereafter construct an analytic function. For instance taking $u(x, y) = \cosh x \cos y$ it follows

$$\frac{\partial u}{\partial x} = \cos y \sinh x = \frac{\partial v}{\partial y} \Rightarrow v = \sinh x \int \cos y dy = \sinh x \sin y + \sinh x g(x), \qquad (1.59)$$

$$\frac{\partial u}{\partial y} = -\sin y \cosh x = -\frac{\partial v}{\partial x} \Rightarrow v = \sin y \int \cosh x dx = \sinh x \sin y + \sinh y h(y), \qquad (1.60)$$

such that $g(x) = h(x) = 0$. The conjugate harmonic function is therefore $v(x, y) = \sinh x \sin y$, such $f(x, y) = \cosh x \cos y + i \sinh x \sin y$ is an analytic function by corollary 1.

1.3 Mappings and Transformations

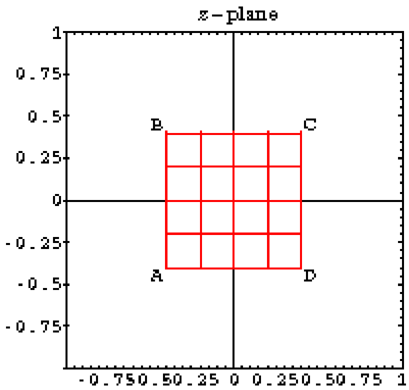
In (1.29) we defined how to map a point $z_0 \in \mathbb{C}$ by means of a function f from the z-plane to the w-plane. Let us now consider some more geometric aspects of this procedure and see how one can visualize functions of complex variables.

1.3.1 Translations

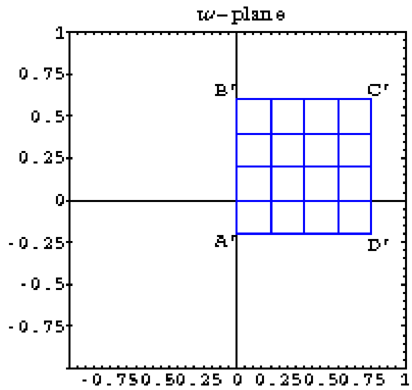
We start with a simple example and consider the function which translates each point in the complex plane by a constant amount $\Delta = \Delta_x + i\Delta_y$

$$w = f_T^{\Delta}(z) := z + \Delta = (x + \Delta_x) + i(y + \Delta_y). \qquad (1.61)$$

We give this map here the name f_T^{Δ}, indicating with the subscript T that we perform a translation and with the superscript Δ the amount of the shift. We depict the map for $\Delta = 0.4 + 0.2i$



$$f_T : z \mapsto w$$
$$f_T(z) = z + 0.4 + 0.2i$$



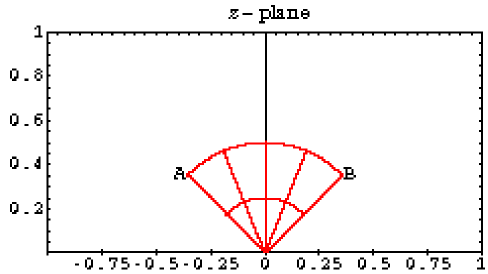
Notice the two regions are of the same shape, size and orientation.

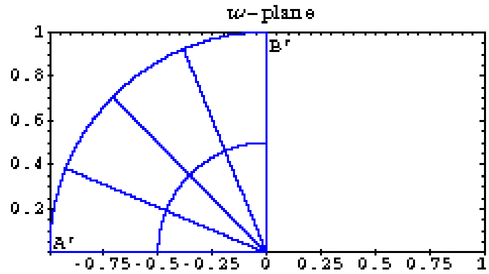
1.3.2 Rotations

Next we consider a rotation, which is most conveniently handled in polar coordinates. Consider

$$w = f_R^{z_0}(z) := zz_0 = re^{i\theta} r_0 e^{i\theta_0} = (rr_0)e^{i(\theta+\theta_0)} \qquad (1.62)$$

Again we introduced some new notation $f_R^{z_0}$, indicating with the subscript R that we perform a rotation and with the superscript z_0 the amount it. As an example we take $z_0 = 2e^{i\pi/4}$, i.e. $r_0 = 2$, $\theta_0 = \pi/4$ and depict $f_R(z) = 2re^{i(\theta+\pi/4)}$ as





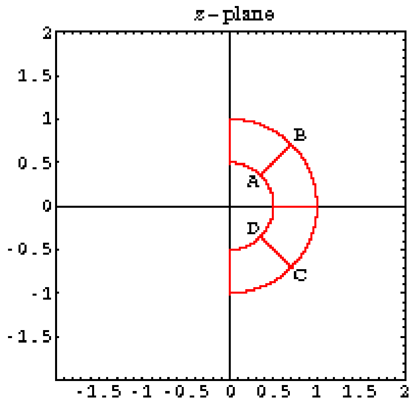
We observe two things: Firstly the radius has been scaled by a factor r_0 and secondly the argument has been increased by θ_0, with the effect that the region has been rotated. Thus strictly speaking $f_R^{z_0}(z)$ as defined in (1.62) is a rotation together with a simultaneous scaling and is only a "pure rotation" for $r_0 = 1$. Nonetheless, due to its simple form, usually the general map $f_R^{z_0}(z)$ is referred to as rotation.

1.3.3 Inversions

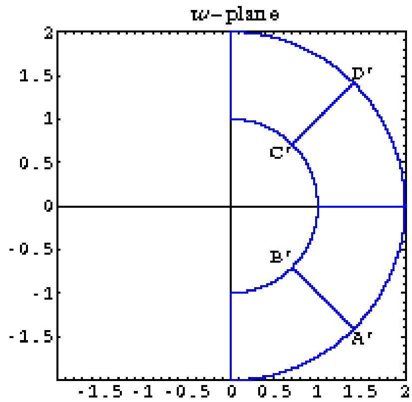
Next we consider the inversion map defined as

$$w = f_I(z) := \frac{1}{z} = \frac{1}{re^{i\theta}} = r^{-1}e^{-i\theta} \tag{1.63}$$

We depict this as



$$f : z \mapsto w$$
$$f_I(z) = 1/z$$



We observe that the interior of the unit circle is mapped into the exterior and vice versa. Let us see what happens when we use Cartesian coordinates instead. Using formula (1.9) we find

$$w = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = u(x,y) + iv(x,y) \tag{1.64}$$

and for the inverse function

$$z = \frac{1}{w} = \frac{\bar{w}}{w\bar{w}} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2} = x(u,v) + iy(u,v). \tag{1.65}$$

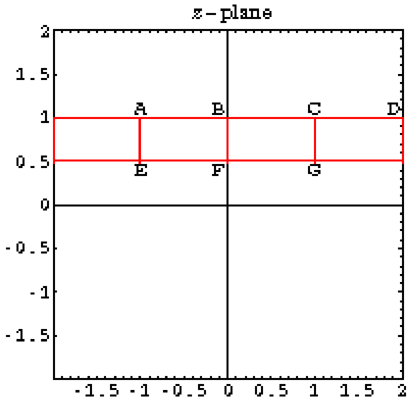
Thus it follows from (1.65) that a horizontal line in the z-plane at $y = c = const$ has to satisfy

$$y = -\frac{v}{u^2+v^2} = c \tag{1.66}$$

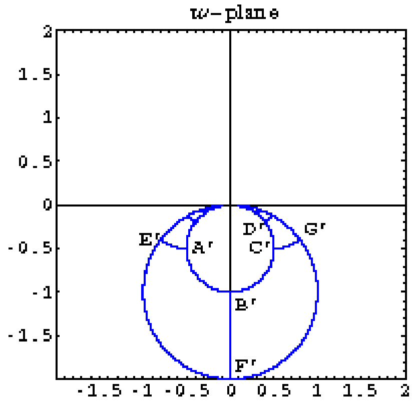
and therefore when completing the square this gives

$$u^2 + v^2 + \frac{v}{c} = 0 \quad \Longleftrightarrow u^2 + \left(v + \frac{1}{2c}\right)^2 = \frac{1}{(2c)^2}. \tag{1.67}$$

The last equation in (1.67) represents a circle with radius $1/2c$ and centre $(0, -1/2c)$ in the w-plane. Thus horizontal lines in the z-plane are mapped into circles in the w-plane. We depict this as



$$f : z \mapsto w$$
$$f_I(z) = 1/z$$



In fact this is true in general and we can formulate this observation in a more general form:
Lemma 1: *The inversion map $w = 1/z$ maps circles and lines always into circles and lines.*

Proof : The equation

$$a(x^2 + y^2) + bx + cy + d = 0 \tag{1.68}$$

parameterizes any circle or line, for $a = 0$, in the z-plane (see e.g. MA1607 Geometry & Vectors). Substituting x, y from (1.65) into (1.68), i.e. $x = u/(u^2 + v^2)$ and $y = -v/(u^2 + v^2)$ gives

$$a + bu - cv + d(u^2 + v^2) = 0. \tag{1.69}$$

Thus for $a = 0$ and $d \neq 0$ any line is mapped into a circle, which we already saw for the special case of a horizontal line. When $a \neq 0$ and $d \neq 0$ circles are mapped into circles. For $a \neq 0$ and $d = 0$ circles are mapped into lines. When $a = 0$ and $d = 0$ lines are mapped into a lines and when $d = 0$ circles are mapped into a lines. $\square$

Let us now organize some of these possible mappings by introducing a new concept.

1.3.4 Conformal mappings

Definition: *A map which preserves the angles between a pair of two intersecting lines is called a* <u>*conformal*</u> *map.*

In fact all previous examples were of this type.

Theorem 4: *Any analytic function $f(z)$ defined on some domain $D \subset \mathbb{C}$ is conformal at the point $z_0 \in D$, if $f'(z_0) \neq 0$.*

Proof : Assuming that f is analytic at the point z_0 we can write with definition (1.41)

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \neq 0. \tag{1.70}$$

In polar form this reads

$$f'(z_0) = \tilde{r}_0 e^{i\tilde{\theta}_0}, \tag{1.71}$$

where by definition

$$\tilde{r}_0 = \left|f'(z_0)\right| \qquad \text{and} \qquad \tilde{\theta}_0 = \arg\left(\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta z\to 0}\left(\arg\frac{\Delta w}{\Delta z}\right). \tag{1.72}$$

In the last equality we used the fact that the limit $\lim_{\Delta z\to 0}$ exists and we can therefore exchange the operations arg and $\lim_{\Delta z\to 0}$. Now we can write $\Delta w = \Delta z \Delta w/\Delta z$ and therefore

$$\arg\Delta w = \arg\Delta z + \arg\frac{\Delta w}{\Delta z}. \tag{1.73}$$

Taking the limit $\Delta z \to 0$ the expression $\arg\Delta z$ just becomes the angle θ_z between the tangent at z_0 and the horizontal on the curve C_z, which is obtained for varying z and the horizontal. Since $\Delta z \to 0$ means that also $\Delta w \to 0$, the expression $\arg\Delta w$ just becomes in this limit the angle θ_w between the tangent at w_0 on the curve C_w and the horizontal. Therefore, taking the limit of the expression in (1.73) yields

$$\theta_w = \theta_z + \tilde{\theta}_0. \tag{1.74}$$

Next we consider two intersecting curves in the z-plane C_z^1, C_z^2 with corresponding images C_w^1, C_w^2 in the w-plane. According to (1.74) the angles at these curves are related as

$$\theta_w^1 = \theta_z^1 + \tilde{\theta}_0 \qquad \text{and} \qquad \theta_w^2 = \theta_z^2 + \tilde{\theta}_0, \tag{1.75}$$

respectively. Notice that $\tilde{\theta}_0$ as defined in (1.72) just involves some generic distances and is therefore the same on both curves C_w^1 and C_w^2. The angles between the tangents are then interpreted as the angles between the curves. Hence the corresponding angles in the z-plane and w-plane are the same

$$\theta_w^1 - \theta_w^2 = \theta_z^1 - \theta_z^2, \tag{1.76}$$

which follows from subtracting the two equations (1.75) from each other. This proves the theorem.□

1.3.5 The linear fractional transformation and the group Gl(2,$\mathbb{C}$)

Definition: *The transformation*

$$\boxed{w = T(z) = \tfrac{az+b}{cz+d} \qquad \text{for } ad - bc \neq 0; a, b, c, d \in \mathbb{C}} \tag{1.77}$$

is called the <u>*linear fractional transformation*</u>.

Notice that since $T'(z) = (ad - bc)/(d + cz)^2$ the restriction $ad - bc \neq 0$ takes care of the restriction in theorem 4 and ensures that $T(z)$ does not become a constant. Also note that the four complex numbers do not determine the map $T(z)$ uniquely, as we may rescale these numbers without changing the map. For instance take $\kappa \in \mathbb{C}$

$$T(z) = \frac{az+b}{cz+d} = \frac{a/\kappa z + b/\kappa}{c/\kappa z + d/\kappa} = \frac{a'z+b'}{c'z+d'}, \tag{1.78}$$

then obviously the numbers a, b, c, d define the same map as a', b', c', d'.

We already discussed some special cases of this. The translation by $\Delta = b$ simply corresponds to $a = 1$, $c = 0$ and $d = 1$. The rotation is obtained by $b = 0$, $c = 0$ and $d = 1$. The inversion map corresponds to $a = 0$, $b = 1$, $c = 1$ and $d = 0$.

One can show that the set of all linear fractional transformations forms a *group*, which is called $Gl(2,\mathbb{C})$ the set of invertible 2×2-matrices with complex entries. In general a groups is defined in the following abstract sense:

Definition: *A group $(g,\circ)$ is a set of elements equipped with a binary operation $\circ$, satisfying the following:*

i) Closure: For any two elements $a, b \in g$ also $a \circ b \in g$.

ii) Existence of the identity: For all elements $a \in g$ there exists an elements $e \in g$, such that $e \circ a = a \circ e = a$.

iii) Existence of the inverse: For each elements $a \in g$ there exists an element $a^{-1} \in g$, such that $a^{-1} \circ a = a \circ a^{-1} = e$.

iv) Associativity: For any three elements $a, b, c \in g$ the relation $(a \circ b) \circ c = a \circ (b \circ c)$ is satisfied.

In order to establish that the linear fractional transformations indeed form a group in the above specified sense we have to show that:
i) Any succession (composition) of two linear fractional transformation $w = (T_1 \circ T_2)(z)$ is again a linear fractional transformation. We compute for this

$$\frac{a_1 z + b_1}{c_1 z + d_1} \circ \frac{a_2 z + b_2}{c_2 z + d_2} = \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)}, \tag{1.79}$$

which is again a linear fractional transformation of the form (1.77)

$$T_3(z) = \frac{a_3 z + b_3}{c_3 z + d_3}, \tag{1.80}$$

with $a_3 = a_1 a_2 + b_1 c_2$ etc.
ii) The inverse $T^{-1}(z)$ is again a linear fractional transformation. From $z = (aw+b)/(cw+d)$ we find

$$T^{-1}(z) = \frac{dz - b}{-cz + a}. \tag{1.81}$$

iii) A unit element exists, i.e. $z = T(z)$.
iv) The composition of three linear fractional transformations is associative, i.e. $T_1 \circ (T_2 \circ T_3) = (T_1 \circ T_2) \circ T_3$. We leave this as an exercise.
Instead of thinking about the group as being represented by the above specified maps we may also represented it by 2×2-matrices via the following correspondence

$$T(z) \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.82}$$

Then the composition of two maps just becomes a matrix multiplication

$$(T_1 \circ T_2)(z) \rightsquigarrow \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

which produces the same values for a_3, b_3, c_3 and d_3.
We will now establish that lemma 1 can be extended from the inversion map to the entire linear fractional transformation.
Lemma 2: *The linear fractional transformation $w = T(z)$ maps circles and lines always into circles and lines.*

Proof : For $c = 0$ the linear fractional transformation $T(z)$ can be written as successive rotation and a translation

$$T^{c=0}(z) = \frac{a}{d}z + \frac{b}{d} = f_T^{b/d} \circ f_R^{a/d}(z), \tag{1.83}$$

which has the stated properties. Taking now $c \neq 0$ we can write

$$T^{c\neq 0}(z) = \frac{az+b}{c(z+d/c)} = \frac{a(z+d/c) - ad/c + b}{c(z+d/c)} = \frac{a}{c} + \frac{b - ad/c}{cz+d} \tag{1.84}$$

$$= f_T^{a/c} \circ f_R^{(bc-ad)/c} \circ f_I \circ f_T^{d} \circ f_R^{c}(z). \tag{1.85}$$

This proves the lemma, as it was shown above that all these individual transformations map circles and lines into circles and lines.□

Alternatively we could have started with the parameterization (1.68) as in the proof for lemma 1 and subsequently used the linear fractional transformation to verify that the resulting equation is of similar type. (exercise)
Theorem 5: *The linear fractional transformation $w = T(z)$ maps three distinct points z_1, z_2, z_3 uniquely into three distinct points w_1, w_2, w_3. The map is determined by the equation*

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}. \tag{1.86}$$

Proof : First we bring $w = T(z)$ into the form of a bilinear transformation

$$cwz + dw - az - b = 0, \tag{1.87}$$

which is linear in z, linear in w and bilinear in wz. (For this reason the linear fractional transformation is sometimes also called the bilinear transformation.) by introducing some new abbreviations we re-write (1.87) as

$$wz + \alpha z + \beta w + \gamma = 0, \qquad \text{with } \alpha = -\frac{a}{c}, \beta = \frac{d}{c}, \gamma = -\frac{b}{c}. \tag{1.88}$$

Now if $w_i = T(z_i)$ for $i = 1, 2, 3$ we can write this set of three equations as

$$M \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} z_1 & w_1 & 1 \\ z_2 & w_2 & 1 \\ z_3 & w_3 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = -\begin{pmatrix} w_1 z_1 \\ w_2 z_2 \\ w_3 z_3 \end{pmatrix}. \tag{1.89}$$

When $\det M \neq 0$ we can solve this uniquely for α, β, γ as

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = -M^{-1} \begin{pmatrix} w_1 z_1 \\ w_2 z_2 \\ w_3 z_3 \end{pmatrix} \tag{1.90}$$

where we evaluate the inverse of M to

$$M^{-1} = -\frac{1}{\det M} \begin{pmatrix} w_{23} & w_{31} & w_{12} \\ z_{32} & z_{31} & z_{21} \\ (w_3 z_2 - w_2 z_3) & (w_1 z_3 - w_3 z_1) & (w_2 z_1 - w_1 z_2) \end{pmatrix}, \tag{1.91}$$

$$\det M = w_1 z_{32} + w_2 z_{13} + w_3 z_{21} = z_1 w_{23} + z_2 w_{31} + z_3 w_{12}. \tag{1.92}$$

We denoted the differences $w_{ij} := w_i - w_j$ and $z_{ij} := z_i - z_j$ for $i, j = 1, 2, 3$. Therefore when evaluating (1.90) we obtain

$$\alpha = (w_2 w_3 z_{23} + w_1 w_2 z_{13} + w_1 w_3 z_{31}) / \det M, \tag{1.93}$$

$$\beta = (z_1 w_1 z_{32} + z_2 w_2 z_{13} + z_3 w_3 z_{21}) / \det M, \tag{1.94}$$

$$\gamma = (z_1 w_2 w_3 z_{32} + z_2 w_1 w_3 z_{13} + z_3 w_1 w_2 z_{21}) / \det M. \tag{1.95}$$

Substituting the expressions for α, β, γ back into (1.88) yields (1.86), after some computation that is. Clearly when $\det M \neq 0$ the solution for α, β, γ is unique and therefore also the map $T(z)$.

What is left, is to see what happens in the case $\det M = 0$. For this situation we write (1.92) in three different versions

$$w_{23} z_1 + z_{32} w_1 + w_3 z_2 - w_2 z_3 = 0, \tag{1.96}$$

$$w_{31} z_2 + z_{13} w_2 + w_1 z_3 - w_3 z_1 = 0, \tag{1.97}$$

$$w_{12} z_3 + z_{21} w_3 + w_2 z_1 - w_1 z_2 = 0, \tag{1.98}$$

and compare this with (1.87). We can now interpret (1.96) as an equation in z_1 and w_1, (1.97) as an equation in z_2 and w_2 and (1.98) as an equation in z_3 and w_3. From the comparison we observe that $c = 0$ and that $a = w_{32} = w_{13} = w_{21}$, which gives

$$\left.\begin{array}{l} w_2 = w_3 - a \\ w_1 = w_3 + a \end{array}\right\} \Rightarrow w_{21} = -2a \neq a \quad \text{for} \;\; a \neq 0. \tag{1.99}$$

Hence, for distinct points z_1, z_2, z_3 and w_1, w_2, w_3 there is no solution to $\det M = 0$. The only solution was the unique one we computed above and the theorem is therefore proven.$\square$

Example 1: Determine the linear fractional transformation $w = T(z)$, which maps the points $z_1 = i, z_2 = 2, z_3 = -i$ uniquely onto $w_1 = -1, w_2 = 0, w_3 = 1$.

Solution : We substitute these points into (1.86)

$$\frac{(w+1)(0-1)}{(w-1)(0+1)} = \frac{(z-i)(2+i)}{(z+i)(2-i)}, \tag{1.100}$$

and solve the resulting equation for w. It is useful to introduce a few auxiliary quantities to facilitate these type of computations. For instance, for (1.100) we have

$$-\frac{w+1}{w-1} = \frac{2z+iz-2i+1}{2z-iz+2i+1} = \frac{2z+1+i(z-2)}{2z+1-i(z-2)} = \frac{p}{q} = P, \tag{1.101}$$

where we intruduced the quantities $p := 2z+1+i(z-2)$, $q := 2z+1-i(z-2)$ and $P = p/q$. We then find from (1.101)

$$w+1 = P - wP, \tag{1.102}$$

such that

$$w = \frac{P-1}{P+1} = \frac{p/q-1}{p/q+1} = \frac{p-q}{p+q}. \tag{1.103}$$

Therefore

$$w = T(z) = \frac{iz-2i}{2z+1}. \tag{1.104}$$

As pointed out above (1.78) the answer is not unique as we can multiply the nominator and denominator by any complex number $\kappa \neq 0$. It is also useful to verify quickly that no mistake has been made in the algebra by checking that the points z_1, z_2, z_3 are indeed mapped onto w_1, w_2, w_3. For instance we compute $T(z_1) = (-1-2i)/(2i+1) = -1 = w_1$.

Example 2: Determine the linear fractional transformation $w = T(z)$, which maps the points $z_1 \to \infty, z_2 = i, z_3 = 0$ uniquely onto $w_1 = 2, w_2 = i, w_3 \to \infty$.

Solution : When infinity is involved in either of the points carry out first that limit in (1.86) on the left and right hand sides where infinity occurs. For the case at hand we have to compute the limit on both sides of the equation (1.86)

$$\lim_{w_3\to\infty} \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(w-w_1)}{(w_2-w_1)} \lim_{w_3\to\infty} \frac{(w_2-w_3)}{(w-w_3)} \tag{1.105}$$

$$= \frac{(w-w_1)}{(w_2-w_1)} \lim_{w_3\to\infty} \frac{\frac{d}{dw_3}(w_2-w_3)}{\frac{d}{dw_3}(w-w_3)} = \frac{w-w_1}{w_2-w_1} \tag{1.106}$$

$$= \lim_{z_1\to\infty} \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{z_2-z_3}{z-z_3}. \tag{1.107}$$

We used L'Haospitals rule in (1.106). Subsequently we substitute the remaining points into (1.107)

$$\frac{(w-2)}{(i-2)} = \frac{(i-0)}{(z-0)}. \tag{1.108}$$

Solve then for w

$$w = T(z) = \frac{2z-(1+2i)}{z}. \tag{1.109}$$

In the previous example we have added "the point" at infinity in order to achieve a proper one-to-one mapping. This has a name:

Definition: *The extended complex plane $\mathbb{C}^*$ is the complex plane plus infinity, i.e. one denotes $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$.*

Definition: *The cross ratio (z_1, z_2, z_3, z_4) is the image of z_4 which maps the points (z_1, z_2, z_3) onto $(0, 1, \infty)$*

$$T_c(z_4) = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)}. \tag{1.110}$$

The expression (1.110) follows from (1.86).

Theorem 6: *The cross ratio (z_1, z_2, z_3, z_4) is real, i.e.*

$$\arg T_c(z_4) = \arg \frac{(z_4 - z_1)}{(z_4 - z_3)} - \arg \frac{(z_2 - z_1)}{(z_2 - z_3)} = 0 \text{ or } \pi, \tag{1.111}$$

if and only if the four points z_1, z_2, z_3, z_4 lie on a line or a circle.

Proof : When $T_c(z_4) \in \mathbb{R}$ the entire image of the points (z_1, z_2, z_3, z_4) is on the real axis. Therefore we know from lemma 2 that (z_1, z_2, z_3, z_4) must have been on a line or circle. When $T_c(z_4) \notin \mathbb{R}$ the image of the points (z_1, z_2, z_3, z_4) is not a line and certainly not a circle. Therefore we know from lemma 2 that (z_1, z_2, z_3, z_4) can not have been on a line or a circle.$\square$

Example 3: Compute the cross ratio $(2, -2, 2i, z_4)$ and use theorem 6 to decide whether the points $\hat{z}_{1/2} = 1 \pm i\sqrt{3}$ and $\hat{z}_{3/4} = 2 \pm i$ lie on the circle $|z| = 2$.

Solution : The cross ratio is computed to

$$T_c(z_4) = \frac{(z_4 - 2)(-2 - 2i)}{(z_4 - 2i)(-2 - 2)} = \frac{(z_4 - 2)(1 + i)}{2(z_4 - 2i)}. \tag{1.112}$$

We then evaluate

$$T_c(1 + i\sqrt{3}) = -\frac{1}{2}(1 + \sqrt{3}) \in \mathbb{R}, \quad T_c(1 - i\sqrt{3}) = (1 + \sqrt{3})^{-1} \in \mathbb{R}, \tag{1.113}$$

$$T_c(2 + i) = \frac{1}{10}(i - 3) \notin \mathbb{R}, \quad T_c(2 - i) = \frac{1}{26}(i + 5) \notin \mathbb{R}. \tag{1.114}$$

We conclude that the points $\hat{z}_{1/2}$ are situated on the circle $|z| = 2$, whereas the points $\hat{z}_{3/4}$ are not. Of course we can reach the same conclusion by representing these numbers in the polar form (1.20). Then it follows immediately, with $|z| = \sqrt{x^2 + y^2}$ for $z = x + iy$, that. $\left|\hat{z}_{1/2}\right| = 2$ and $\left|\hat{z}_{3/4}\right| = \sqrt{5}$.

Example 4: Determine the image of the unit circle $|z| = 1$, which is mapped by

$$w = f(z) = (2 - i) - 2iz \tag{1.115}$$

into the w-plane.

Solution : From theorem 5 we know that the map is uniquely determined by mapping three distinct points. Let us take the points $z_1 = 1, z_2 = -1, z_3 = i$ and compute their images

$$w_1 = 2 - 3i, \qquad w_2 = 2 + i, \qquad w_3 = 4 - i. \tag{1.116}$$

Assume first that the image is a line, i.e. of the form $v = \alpha u + c$. These three points can not lie on a line, which can be argued as follows. The line through w_1 and w_2 is obviously a vertical line at $x = 2$, but since $\mathrm{Re}\, w_3 = 4$ it can not be located at the same line. Since the image is not a line it follows by lemma 2 that is has to be a circle. Any circle can be parameterized by

$$(u - u_0)^2 + (v - v_0)^2 = r^2. \tag{1.117}$$

Substituting the three points w_i for $i = 1, 2, 3$ into (1.117) gives

$$w_1 : \qquad (2 - u_0)^2 + (3 + v_0)^2 = r^2, \tag{1.118}$$

$$w_2 : \qquad (2 - u_0)^2 + (1 - v_0)^2 = r^2, \tag{1.119}$$

$$w_3 : \qquad (4 - u_0)^2 + (1 + v_0)^2 = r^2. \tag{1.120}$$

Thus we have three equation for the three unknown u_0, v_0, r. Let us solve these equations

$$\left.\begin{array}{l} (1.119) - (1.118): \;\; (1 - v_0)^2 - (3 + v_0)^2 = 0 \Leftrightarrow -8 - 8v_0 = 0 \Rightarrow v_0 = -1 \\ (1.119) - (1.120): (2 - u_0)^2 - (4 - u_0)^2 + 4 = 0 \Leftrightarrow 4u_0 - 8 = 0 \Rightarrow u_0 = 2 \end{array}\right\} \Rightarrow r = 2.$$

This means the image of the unit circle is a circle centered at $(2, -1)$ of radius 2

$$(u - 2)^2 + (v + 1)^2 = 4. \tag{1.121}$$

In fact noting that $f(z)$ can be decomposed as $f(z) = f_T^{2-i} \circ f_R^{-2i}(z)$ this conclusion can be drawn more directly. From this follows that f_R^{-2i} multiplies the radius by a factor 2 and f_T^{2-i} will move the centre from $(0, 0)$ to $(2, -1)$.

1.4 Branch points, branch cuts and Riemann surfaces

We already saw in section 1.2.1. that functions can be multi-valued. We will now see how we can make them *single valued* and *analytic*. Our prime example will be the logarithmic function, from which the behaviour of many other functions can be derived.

1.4.1 The logarithmic function

We have already encountered, see (1.34), the multi-valued function

$$f_n(z) = \ln z = \ln(re^{i\theta}) = \ln r + i\theta + 2\pi i n \qquad \text{with } n \in \mathbb{Z}. \tag{1.122}$$

In order to make this function single valued we could simple choose one value of n, for instance $n = 0$ and consider the function

$$F(z) = f_0(z) = \ln r + i\theta \qquad \text{for } r \in \mathbb{R}^+, -\pi < \theta \leq \pi. \tag{1.123}$$

We have made our function single valued, but we still have the problem that the function is not analytic, since it is not continuous when we cross the negative real axis. Crossing from the upper half to the lower half z-plane the θ-value jumps from π to $-\pi$, i.e. taking the value $\theta = \pi$ we find the value $\theta = -\pi$ in any small neighbourhood around a point on the negative real axis. This means the derivatives of this function do not exist on the negative real axis. We may remedy this by cutting out a ray from the domain and define instead of (1.123) the function

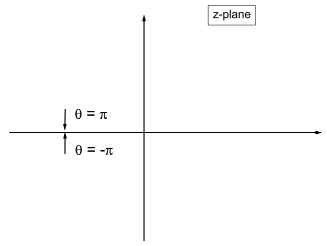


Figure 5: The discontinuity of lnz.

$$f_0^{\tilde{\theta}}(z) = \ln z = \ln r + i\theta \qquad \text{for } r \in \mathbb{R}^+, \tilde{\theta} < \theta < \tilde{\theta} + 2\pi. \tag{1.124}$$

Now this function is single valued and continuous in the entire complex plane from which the ray at $\theta = \tilde{\theta}$ has been taken away. To verify that the function is indeed analytic, let us check whether the Cauchy-Riemann conditions hold and if the partial derivatives are continuous, (see theorem 2). Let us first re-write the Cauchy-Riemann conditions (1.47) in polar coordinates. From (1.19) we find

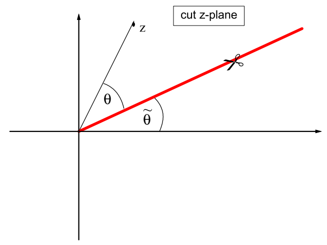


Figure 6: The z-plane with a cut.

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial u}{\partial r}\frac{1}{\cos\theta} \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial \theta}\frac{1}{r\cos\theta} \end{aligned}\right\} \Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}, \tag{1.125}$$

and

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial \theta}\frac{1}{r\cos\theta} \\ -\frac{\partial v}{\partial x} &= -\frac{\partial v}{\partial r}\frac{\partial r}{\partial x} = -\frac{\partial v}{\partial r}\frac{1}{\cos\theta} \end{aligned}\right\} \Rightarrow \boxed{\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}}. \tag{1.126}$$

Now we apply (1.125) and (1.126) to the function $f_0^{\tilde{\theta}}(z)$ in (1.124), i.e. for $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$. We compute

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial r} \tag{1.127}$$

for all $r \in \mathbb{R}^+, \tilde{\theta} < \theta < \tilde{\theta} + 2\pi$. Thus the Cauchy-Riemann conditions are indeed satisfied and since all partial derivatives are continuous the function $f_0^{\tilde{\theta}}(z)$ is an analytic function by theorem 2. Note that this would not be the case without the cut.

We make these notions a bit more formal:

Definition: *A* <u>branch</u> *$F(z)$ of a multi-valued function $f(z)$ is any single valued function, which is analytic in some domain $D \subset \mathbb{C}$, where $F(z_0) = f(z_0)$ for all $z_0 \in D$.*

Definition: *A* <u>branch cut</u> *is a curve in the complex plane across which an analytic multivalued function is discontinuous.*

Definition: *A point which is shared by all branches of the function is called a* <u>branch point</u>.

Definition: *The* <u>principal branch</u> *$F^p(z)$ of the logarithmic function is defined as*

$$F^p(z) = f_0^{-\pi}(z) = \ln z = \ln r + i\theta \qquad \text{for } r \in \mathbb{R}^+, -\pi < \theta < \pi. \tag{1.128}$$

The principal branch of other functions are those corresponding to the principle branch of the logarithmic function.

Note that for many functions $f_n^{\tilde{\theta}}(z)$ we can identify a possible branch for the logarithmic function. The point $z = 0$ is common to all branches and therefore this is the branch point of the logarithmic function.

1.4.2 Riemann surfaces

Branch cuts are one way to handle discontinuities of multivalued functions in the complex plane. However, for some applications one does not wish to restrict the values of θ. An alternative method which avoids this limitation is to use Riemann surfaces.

Definition: *A* <u>*Riemann surface*</u> *is a surface-like configuration that covers the complex plane with several, often infinitely many, "sheets." Functions defined on Riemann surfaces can be made single valued and continuous for the entire range of θ.*

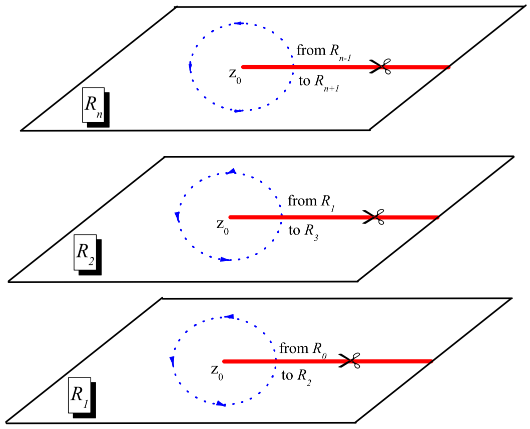


Figure 7: Riemann surface for lnz.

Let us now see how to construct such surfaces for the example for the logarithmic function.

- We consider the z-plane and cut it along a branch cut on the positive real axis, such that θ is allowed to take the values $0 \leq \theta < 2\pi$. We denote this plane as $\mathcal{R}_1$.

- Next we cut another z-plane in the same way and denote it as $\mathcal{R}_2$. We place this sheet on top of $\mathcal{R}_1$ and identify the lower edge of the cut in $\mathcal{R}_1$ with the upper edge of the cut in $\mathcal{R}_2$. In this second Riemann sheet θ takes on the values $2\pi \leq \theta < 4\pi$.

- We proceed further in the same way and identify next the lower edge of the cut in $\mathcal{R}_2$ with the upper edge of the cut in the new sheet $\mathcal{R}_3$ which is placed on top of $\mathcal{R}_2$. In this third Riemann sheet θ takes on the values $4\pi \leq \theta < 6\pi$.

- One may continue this procedure up to infinity. We can also allow negative values for θ by continuing in the other direction, i.e. we identify the upper edge in the cut of $\mathcal{R}_1$ with lower edge of a branch cut in a plane $\mathcal{R}_0$ which is placed below $\mathcal{R}_1$ and in which θ can now take on the values $-2\pi \leq \theta < 0$.

- Preceding this way we can achieve that $\theta \in \mathbb{R}$ without any restriction.

This means we have the Riemann sheets

$$\mathcal{R}_1 = \{r, \theta : r \in \mathbb{R}^+, 0 \leq \theta < 2\pi\}, \tag{1.129}$$
$$\mathcal{R}_2 = \{r, \theta : r \in \mathbb{R}^+, 2\pi \leq \theta < 4\pi\}, \tag{1.130}$$
$$\vdots$$
$$\mathcal{R}_n = \{r, \theta : r \in \mathbb{R}^+, 2(n-1)\pi \leq \theta < 2n\pi\} \quad \text{for } n \in \mathbb{Z}. \tag{1.131}$$

The logarithm is now a single valued and continuous function on the Riemann surface

$$D = \bigcup_{n=-\infty}^{\infty} \mathcal{R}_n. \tag{1.132}$$

Therefore in this way of looking at the problem the logarithm becomes an analytic function everywhere except at the origin.

1.4.3 Roots and other irrational functions

We may now understand the behaviour of other multivalued functions by employing the knowledge we have accumulated by studying the logarithmic function.

For instance we can now handle functions of the type $z^{1/n}$ with $n \in \mathbb{Z}$.

The function $\sqrt{z}$ We consider first the square root function. Initially we assume that we could have infinitely many branches just like for the logarithmic function

$$f_n(z) = \sqrt{z} = \exp\left(\frac{1}{2}\ln z\right) = \exp\left[\frac{1}{2}(\ln r + i\theta + 2\pi i n)\right] = \sqrt{r}e^{i\frac{\theta + n2\pi}{2}}. \tag{1.133}$$

However, there are only two branches which differ from each other, namely $f_0(z)$ and $f_1(z)$. Note that for other values of n we do not produce new functions, as for instance $f_2(z) = f_0(z)$, $f_3(z) = f_1(z)$, etc. According to (1.128) it is clear that the principle branch of this function, which corresponds to the one of the logarithmic function is

$$F^p(z) = f_0(z) = \sqrt{r}e^{i\theta/2} \tag{1.134}$$

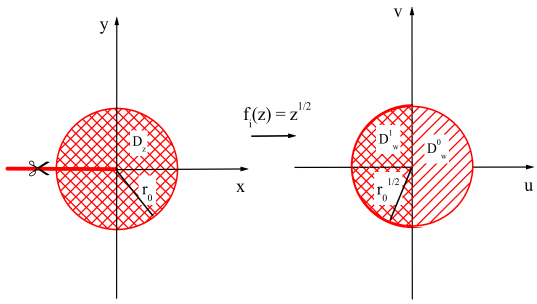


Figure 8: Different branches of the square root.

for $r \in \mathbb{R}^+, -\pi < \theta < \pi$. Let us see how certain regions in the z-plane are mapped to the w-palne. For instance, if we restrict the values of r and take the sliced disk of radius r_0

$$D_z = \{r, \theta : r < r_0, \quad -\pi < \theta < \pi\}, \tag{1.135}$$

the principle branch of the square root $f_0(z)$ maps this sliced disk onto the half disk of radius $\sqrt{r_0}$ in the right half plane

$$D_w^0 = \{r, \theta : r < \sqrt{r_0}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}. \tag{1.136}$$

We can check a few values on the boundary: $f_0(i) = e^{i\pi 5/4}$, $f_0(1) = e^{i\pi}$, $f_0(-i) = e^{i\pi 3/4}$, $f_0(e^{i\pi/6}) = e^{i\pi 13/12}$.

Taking instead the other branch $f_1(z)$, we observe that it maps D_z onto the half disk of radius $\sqrt{r_0}$ in the left half plane

$$D_w^1 = \{r, \theta : r < \sqrt{r_0},\ \frac{\pi}{2} < \theta < \frac{3\pi}{2}\}. \tag{1.137}$$

We can check a few values on the boundary: $f_1(i) = e^{i\pi/4}$, $f_1(1) = 1$, $f_1(-i) = e^{-i\pi/4}$, $f_1(e^{i\pi/6}) = e^{i\pi/12}$.

Having determined the branch cuts we can also construct the Riemann surface, which consists in this case of two sheets. For instance we can turn $f(z) = \sqrt{z}$ into a single valued analytic function when we define it on

$$D = \mathcal{R}_1 \cup \mathcal{R}_2 \tag{1.138}$$

with

$$\mathcal{R}_1 = \{r, \theta : r \in \mathbb{R}^+, 4\pi n \leq \theta < 4\pi n + 2\pi, n \in \mathbb{Z}\}, \tag{1.139}$$
$$\mathcal{R}_2 = \{r, \theta : r \in \mathbb{R}^+, 4\pi n + 2\pi \leq \theta < 4\pi(n+1), n \in \mathbb{Z}\}. \tag{1.140}$$

Notice that unlike as for the logarithmic function in this case we can always return to the other sheet as the value of $\sqrt{z}$ is the same when we have passed along the two sheets, i.e. when we have increased θ by 4π. To see this more explicitly see figure 9. Following the numbers from 1 to 6 to return to the starting point in the first sheet you have increased the value of θ by 4π.

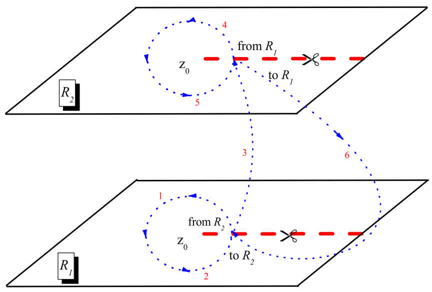


Figure 9: Riemann surface for $f(z) = z^{1/2}$.

The function $z^{1/3}$ Next we consider

$$f_n(z) = z^{1/3} = \exp\left(\frac{1}{3}\ln z\right) = \exp\left[\frac{1}{3}(\ln r + i\theta + 2\pi i n)\right] = r^{1/3} e^{i\frac{\theta + n2\pi}{3}}. \tag{1.141}$$

Now there are three possible branches $f_0(z)$, $f_1(z)$ and $f_2(z)$. Again we use the logarithmic function to identify the principle branch as

$$F^p(z) = f_0(z) = r^{1/3} e^{i\theta/3} = \exp\left(\frac{1}{3}\ln z\right) \qquad \text{for } r \in \mathbb{R}^+, -\pi < \theta < \pi. \tag{1.142}$$

Thus the complex plane which is slit open along the negative real axis

$$D_z = \{r, \theta : r \in \mathbb{R}^+,\ \ -\pi < \theta < \pi\} \tag{1.143}$$

is mapped by the principle branch $f_0(z)$ onto the wedge region

$$D_w^0 = \{r, \theta : r \in \mathbb{R}^+,\ \ -\frac{\pi}{3} < \theta < \frac{\pi}{3}\}, \tag{1.144}$$

whereas the branches $f_1(z)$ and $f_2(z)$ map D_z onto the wedges

$$D_w^1 = \{r, \theta : r \in \mathbb{R}^+, \ \frac{\pi}{3} < \theta < \pi\}, \tag{1.145}$$

$$D_w^2 = \{r, \theta : r \in \mathbb{R}^+, \ \pi < \theta < \frac{5}{3}\pi\}, \tag{1.146}$$

respectively.

The Riemann surface consists in this case of three sheets. For instance we can turn $f(z) = z^{1/3}$ into a single valued analytic function when we define it on

$$D = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \tag{1.147}$$

with

$$\mathcal{R}_1 = \{r, \theta : r \in \mathbb{R}^+, 6\pi n \leq \theta < 6\pi n + 2\pi, n \in \mathbb{Z}\}, \tag{1.148}$$

$$\mathcal{R}_2 = \{r, \theta : r \in \mathbb{R}^+, 6\pi n + 2\pi \leq \theta < 6\pi n + 4\pi, n \in \mathbb{Z}\}, \tag{1.149}$$

$$\mathcal{R}_3 = \{r, \theta : r \in \mathbb{R}^+, 6\pi n + 4\pi \leq \theta < 6\pi(n+1), n \in \mathbb{Z}\}. \tag{1.150}$$

Having seen how to construct the surface for the square root function you may easily draw this surface yourself. (Exercise!)

Let us consider some more explicit cases:

Example 1:

Find at least two different choices of branch cuts for the function

$$f(z) = (z - z_0)^{1/2} \tag{1.151}$$

such that it becomes single valued and analytic. Determine the branch point and describe the corresponding Riemann surfaces.

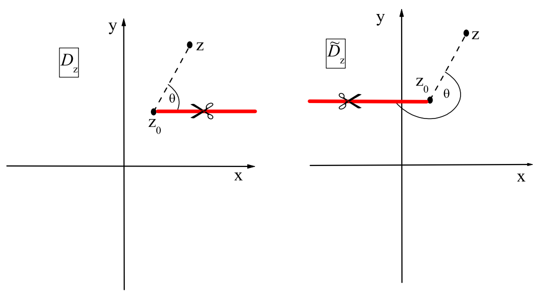


Figure 10: Two choices for branch cuts.

Solution : We can write

$$f(z) = (z - z_0)^{1/2} = \sqrt{\tilde{r}} \exp(i\tilde{\theta}/2) \tag{1.152}$$

and define it on

$$D_z = \{\tilde{r}, \tilde{\theta} : \tilde{r} \in \mathbb{R}^+, \ 0 < \tilde{\theta} < 2\pi\}, \tag{1.153}$$

such that the branch cut extends from the branch point z_0 horizontally to plus infinity. As we already said, this is only a matter of convention and we could also define the function on a different domain, such as

$$\tilde{D}_z = \{\tilde{r}, \tilde{\theta} : \tilde{r} \in \mathbb{R}^+, \ -\pi < \tilde{\theta} < \pi\}, \tag{1.154}$$

achieving just the same effect. In figure 10 the different types of domains of analyticity are depicted. The Riemann surface is just the same as the one for $\sqrt{z}$ with the difference that the origin has been shifted to z_0. The point z_0 is shared by all branches and therefore the branch point.

Example 2: Determine the branch cuts needed to make the function

$$f(z) = (z^2 - 1)^{1/2} \tag{1.155}$$

single valued and analytic.

Solution : We can write

$$\begin{aligned} f(z) &= (z+1)^{1/2}(z-1)^{1/2} = \sqrt{r_1}\exp(i\theta_1/2)\sqrt{r_2}\exp(i\theta_2/2) = f_1(r_1,\theta_1)f_2(r_2,\theta_2) && (1.156) \\ &= \sqrt{r_1 r_2}\exp\left[i(\theta_1+\theta_2)/2\right] && (1.157) \end{aligned}$$

and thus at first we can make the functions f_1, f_2, which only depend separately on (r_1, θ_1) or on (r_2, θ_2), individually single valued and analytic.

- In order to make the function single valuedness we have already selected a particular branch in (1.157).
- To achieve analyticity we define the functions for instance both on D_z as defined in (1.153), by replacing $(\tilde{r}, \tilde{\theta}) \to (r_1, \theta_1)$ and $(\tilde{r}, \tilde{\theta}) \to (r_2, \theta_2)$, respectively. Hence we have two branch cuts which extend from the branch points $z = -1$ and $z = 1$ horizontally to plus infinity. Notice that this is convention and other choices would have been possible, such as the principle branch cuts.
- Next we have to settle the question of how several branch cuts effect each other. Let us look at different regions on the real axis.
- The part $z \in (-\infty, -1)$ poses no problem as f_1 as well as f_2 are smooth when crossing this line.
- Next we consider the line segment $z \in [-1, 1]$. Here the function f_2 is continuous, but f_1 is not and therefore we have to cut out this line segment.
- Next we consider the part $z \in (1, \infty)$. Above the axis we have $\theta_1 = \theta_2 = 0$, such that $\theta_1 + \theta_2 = 0$ and therefore $f(z) = \sqrt{r_1 r_2}$. Below the axis we have $\theta_1 = \theta_2 = 2\pi$, such that $\theta_1 + \theta_2 = 4\pi$ and we therefore also have $f(z) = \sqrt{r_1 r_2}$. This means the function also crosses this part of the axis in a smooth way, despite the fact that the individual functions f_1, f_2 would have branch cuts in that regime.

Overall this means if we take out the line segment $z \in [-1, 1]$ the function $f(z)$ becomes single valued and analytic. We could have also started out by defining the functions f_1, f_2 on other domains than D_z and we would have ended up with a different type of cut or possibly cuts. (For a more detailed discussion of this issue see exercise sheet 3.)

1.5 The Riemann mapping theorem

So far we have mainly concentrated on the question of how certain geometric configurations (boundaries of regions) transform when mapped by analytic functions. Next we want to address the question of how entire regions are mapped by conformal transformations. This sort of problem is characterized by the Riemann mapping theorem:

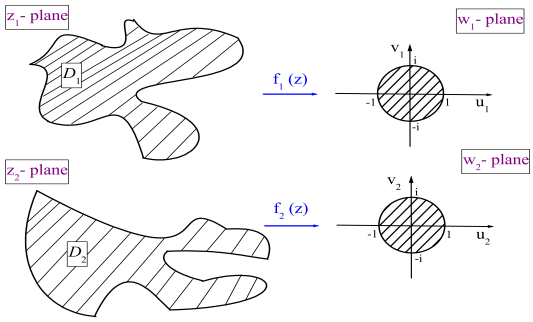


Figure 11: The Riemann mapping theorem

Theorem 7: *(Riemann mapping theorem) Given a simply connected region $D \subset \mathbb{C}$ (i.e. D has no holes) which is not the entire plane and a point $z_0 \in D$. Then there exists an analytic function $f : z \mapsto w$ which maps D one-to-one onto the interior of the unit disk $|w| < 1$. The uniqueness of the map can be achieved with the normalization condition $f(z_0) = 0$ and $f'(z_0) > 0$.*

Proof : omitted here but may be found for instance in L.V. Ahlfors, Complex analysis, MacGraw-Hill, New York, 1979.

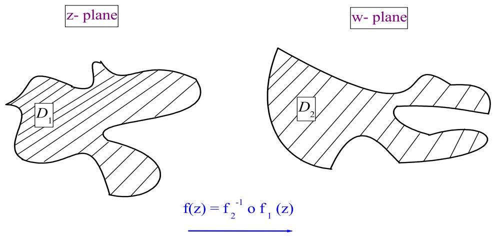


Figure 12: Consequence of theorem 7

An immediate consequence of this theorem is the remarkable fact that *any* two simply connected regions D_1 and D_2 (with the exception that they can not be the entire plane) can be mapped into each other in a one-to-one fashion by a conformal map. From theorem 7 follows that there exist two analytic functions f_1, f_2 which map the simply connected regions D_1,D_2 in a one-to-one fashion to the interior of the unit disk, respectively. See figure 11. Therefore provided the inverse map of f_2 exists we can map directly $D_1 \mapsto D_2$ by

$$f(z) = f_2^{-1} \circ f_1(z). \tag{1.158}$$

The two regions D_1 and D_2 are said to be *conformally equivalent*.

Example 1: Find an analytic function which maps the upper half of the complex plane $\operatorname{Im} z > 0$ onto the interior of the unit disk $|w| < 1$.

Solution : First we note that the boundaries of the two regions have to be mapped onto each other and therefore the image of the part $\operatorname{Im} z > 0$ has to be in the interior of the unit circle. We take distinct point on the boundary, the real axis in this case, and demand that they are mapped onto the circle $|w| = 1$

$$z_1 = -1, z_2 = 0, z_3 = 1 \mapsto w_1 = i, w_2 = -1, w_3 = -i. \tag{1.159}$$

Let us assume this map is a linear fractional transformation, then it follows from (1.86) that

$$\frac{(w-i)(-1+i)}{(w+i)(-1-i)} = \frac{(z+1)(0-1)}{(z-1)(0+1)} \tag{1.160}$$

and therefore upon solving this for w we find

$$w = f(z) = \frac{z-i}{z+i}. \tag{1.161}$$

Let us check the uniqueness. We can take $z_0 = i$ and therefore

$$f(i) = 0 \quad \text{and} \quad f'(z_0 = i) = \left.\frac{2i}{(z_0+i)^2}\right|_{z_0=i} = -\frac{i}{2}. \tag{1.162}$$

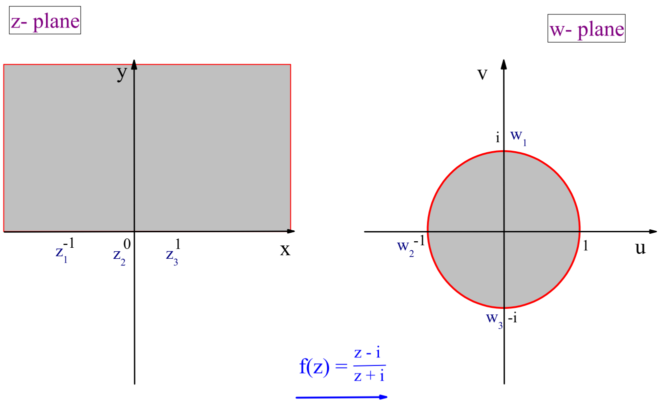


Figure 13: Upper half plane mapped to the unit disk.

The derivative does not quite satisfy the normalization condition specified in the theorem, but with simple rotation we achieve our aim. The function $\tilde{f}(z) = if(z)$ satisfies the requirements and therefore maps the upper half plane uniquely onto the unit disk. Clearly $f(z)$ is also a unique map, albeit with a different normalization condition.

See exercise sheet 3 for more examples.

1.5.1 The Schwarz-Christoffel transformation

We have seen that the linear fractional transformation can be used to map all kinds of exotic regions into each other. Let us see next what kind of images we obtain when we map with non-linear functions, which can even be non-analytic functions in some points. An important example for these type of maps is the Schwarz-Christoffel transformation, which maps the upper half plane onto an n-sided polygon.

In order to see how this works in detail let us recall formula (1.73), which gives a relation between the unit-tangent vector t_z at the point z_0 on the curve C_z and the unit-tangent vector t_w at the image point $w_0 = f(z_0)$ on the curve C_w

$$\arg t_w = \arg t_z + \arg f'(z_0). \tag{1.163}$$

Taking now the curve C_z to run along the real axis in increasing x-direction, we have $\arg t_z = 0$ and therefore

$$\arg t_w = \arg f'(z_0). \tag{1.164}$$

Next we distribute $n-1$ points on the real axis at the positions x_i with $1 \leq i \leq n-1$ in strictly increasing order, that is $x_1 < x_2 < \ldots < x_{n-1}$. We suppose that these points are mapped to $w_i = f(x_i)$ for $1 \leq i \leq n-1$. We introduce an additional point, which is the image of infinity $\lim_{x \to \pm\infty} f(x) = w_n$. We make now the crucial assumption on the form of the derivative of the function $f(z)$

$$f'(z) = c \prod_{i=1}^{n-1} (z - x_i)^{-\mu_i} \qquad c \in \mathbb{C}, -1 < \mu_i < 1. \tag{1.165}$$

Taking the argument of this equation then yields

$$\arg f'(z) = \arg c - \sum_{i=1}^{n-1} \mu_i \arg(z - x_i). \tag{1.166}$$

When z is taken on the x-axis $\arg(x - x_i)$ is a stepfunction

$$\arg(x - x_i) = \begin{cases} \pi & \text{for } x < x_i \\ 0 & \text{for } x > x_i \end{cases}. \tag{1.167}$$

This means when we increase x and pass by the point x_i the derivative $f'(z)$ will change by the amount $\mu_i \pi$. We now interpret the points w_i for $1 \leq i \leq n$ as vertices of a polygon and the values $\mu_i \pi$ as exterior angles at the i-th vertex. The latter interpretation is the explanation for the restriction on the constants μ_i mentioned already in (1.165). Obviously the sum of all exterior angles has to be 2π, which gives the further restriction

$$2\pi = \sum_{i=1}^{n} \mu_i \pi \quad \Rightarrow \quad 2 = \sum_{i=1}^{n} \mu_i. \tag{1.168}$$

We can now understand the next theorem, which goes beyond our previous observations and also ensures under which condition such transformation exist.

Theorem 8: *(Schwarz-Christoffel theorem) Given an n-sided polygon with vertices w_i and exterior angles $\theta_i = \mu_i \pi$ for $1 \leq i \leq n$. Then there exist always n real numbers x_i for $1 \leq i \leq n$ together with a complex constant $c \in \mathbb{C}$ and an analytic function $f : z \mapsto w$ whose derivative is given by*

$$f'(z) = c \prod_{i=1}^{n-1} (z - x_i)^{-\mu_i} \qquad c \in \mathbb{C}, -1 < \mu_i < 1, \tag{1.169}$$

which maps the upper half plane one-to-one onto the interior of the polygon. The points are mapped as $w_i = f(x_i)$ *for* $1 \leq i \leq n-1$ *and* $w_n = \lim_{x\to\pm\infty} f(x)$.

Proof : omitted here, but see before the theorem for justification and plausibility

Let us see how to apply this theorem.

Example 1: Determine the Schwarz-Christoffel transformation, which maps the upper half plane to an equilateral triangle. Map the points $x_1 = 1$ and $x_2 = -1$ to $w_1 = 0$ and $w_2 = a$. Express your result in terms of the quantity

$$\alpha = \int_{-1}^{1} d\hat{z} \frac{1}{(1-\hat{z}^2)^{2/3}} = \sqrt{\pi}\frac{\Gamma(1/3)}{\Gamma(5/6)} \approx 4.20655, \qquad (1.170)$$

where Γ denotes the Gamma function[1].

Solution : The exterior angles θ_i with $i = 1, 2, 3$ for an equilateral triangle are obviously $\theta_i = 2\pi/3$. We can therefore express the derivative (1.169) as

$$f'(z) = c\prod_{i=1}^{2}(z - x_i)^{-2/3}. \qquad (1.171)$$

Let us take next the points x_i to be $x_1 = 1$ and $x_2 = -1$, such that

$$f'(z) = c(z-1)^{-2/3}(z+1)^{-2/3}. \qquad (1.172)$$

Integration (1.169) then yields

$$f(z) = c\int_{1}^{z} d\hat{z}(\hat{z}-1)^{-2/3}(\hat{z}+1)^{-2/3} + \tilde{c}, \qquad (1.173)$$

with $\tilde{c} \in \mathbb{C}$ some integration constant. Let us now fix the constants c and $\tilde{c}$ by substituting the values for all vertices. We have taken here the lower limit to be 1 as this yields simply

$$f(1) = \tilde{c} = w_1 = 0. \qquad (1.174)$$

Furthermore we have

$$f(-1) = c\int_{1}^{-1} d\hat{z}(\hat{z}-1)^{-2/3}(\hat{z}+1)^{-2/3} = w_2, \qquad (1.175)$$

$$\lim_{z\to-\infty} f(z) = c\int_{1}^{-\infty} d\hat{z}(\hat{z}-1)^{-2/3}(\hat{z}+1)^{-2/3} = w_3, \qquad (1.176)$$

$$\lim_{z\to\infty} f(z) = c\int_{1}^{\infty} d\hat{z}(\hat{z}-1)^{-2/3}(\hat{z}+1)^{-2/3} = w_3. \qquad (1.177)$$

[1] The Gamma function can be viewed as a generalization of $n!$ to non-integer values

$$\Gamma(z) := \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\ldots n}{z(z+1)(z+2)\ldots(z+n)} n^z = \int_0^\infty dte^{-t}x^{x-1} \qquad z \neq \mathbb{Z}_0^-.$$

Note that $\Gamma(n+1) = n!$ for $n \in \mathbb{Z}$.

These integrals are quite horrible to solve as they will give hypergeometric functions. However, we do not have to do this since we have more equations than constants left and can express all quantities in terms of the given integral (1.170). Let us take first the equation (1.175)

$$f(-1) = c\int_1^{-1} d\hat{z}(\hat{z}-1)^{-2/3}(\hat{z}+1)^{-2/3} = c\int_1^{-1} d\hat{z}e^{-i\pi 2/3}\frac{1}{(1-\hat{z}^2)^{2/3}} \tag{1.178}$$

$$= ce^{i\pi/3}\int_{-1}^{1} d\hat{z}\frac{1}{(1-\hat{z}^2)^{2/3}} = ce^{i\pi/3}\alpha = w_2 = a. \tag{1.179}$$

In (1.178) we have simply taken out a factor -1 from the bracket $(\hat{z}-1)^{-2/3} = (-1)^{-2/3}(1-\hat{z})^{-2/3}$ with $(-1)^{-2/3} = e^{-i\pi 2/3}$. In the step (1.178) to (1.179) we used $\int_a^b dx = -\int_b^a dx$ with $-1 = e^{i\pi}$. Therefore the constant c is fixed to

$$c = e^{-i\pi/3}a/\alpha. \tag{1.180}$$

Next have to verify that the thirs point lies indeed on the other vertex of the triangle. We therefore evaluate (1.176)

$$w_3 = c\int_1^{-1} d\hat{z}(\hat{z}-1)^{-2/3}(\hat{z}+1)^{-2/3} + c\int_{-1}^{-\infty} d\hat{z}(\hat{z}-1)^{-2/3}(\hat{z}+1)^{-2/3} \tag{1.181}$$

$$= w_2 + ce^{-i\pi 4/3}\int_{-1}^{-\infty} d\hat{z}\,|\hat{z}-1|^{-2/3}\,|\hat{z}+1|^{-2/3} \tag{1.182}$$

$$= w_2 + ce^{-i\pi/3}\int_1^{\infty} d\hat{z}\,|\hat{z}+1|^{-2/3}\,|\hat{z}-1|^{-2/3} \tag{1.183}$$

$$= w_2 + e^{-i\pi/3}w_3 = a + e^{-i\pi/3}w_3. \tag{1.184}$$

In the step from (1.181) to (1.182) we used the fact for the range of the integral $\hat{z}-1 \leq 0$ and $\hat{z}+1 \leq 0$. Introducing the modulus means that we have to take out a minus sign from each factor, i.e. $(-1)^{-2/3}(-1)^{-2/3} = e^{-i\pi 4/3}$. In the step from (1.182) to (1.183) we change the integration variable from $\hat{z} \to -\hat{z}$, leaving the integrant invariant but producing an overall minus sign such that $-e^{-i\pi 4/3} = e^{-i\pi/3}$. In (1.184) we can then drop the modulus and use (1.177) to replace the integral. Solving this for w_3 gives

$$w_3 = \frac{a}{1-e^{-i\pi/3}} = \frac{ae^{-i\pi/3}}{e^{-i\frac{\pi}{3}} - e^{-i\frac{2\pi}{3}}} = \frac{ae^{-i\pi/3}}{e^{-i\frac{\pi}{3}} + e^{i\frac{\pi}{3}}} = \frac{ae^{-i\pi/3}}{2\cos\pi/3} = e^{-i\pi/3}a. \tag{1.185}$$

Thus we obtain indeed an equilateral triangle with vertices $w_1 = 0$, $w_2 = a$ and $w_3 = e^{-i\pi/3}a$.

2. Boundary value problems

We will now see how the material we have learned in the previous section, in particular conformal maps, can be applied in some concrete problems in physics (electrostatic, motion of incompressible fluids, gravity, string theory, ...), biology, chemistry, finance etc. One of the most common and oldest problems are boundary value problems. In general, in a *boundary problem* one is given some information about some unknown function f in some domain D (usually a differential equation) and in addition some information about the function on the boundary ∂D. In case the information at the boundary are

i) the values of the function on the boundary, one speaks of a *Dirichlet problem*

ii) the derivatives of the function one speaks of a *Neumann problem.*

Of course these conditions can also be mixed, such that part of the information consists of the values at the boundary and part of the information consists of the information about the derivatives at the boundary. However, we shall see below that some manipulations are only meaningful when the boundary conditions are either of pure Dirichlet or pure Neumann type. See figure 14.

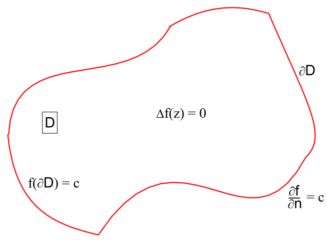


Figure 14: Boundary value problem.

The main solution techniques for boundary value problems are:

i) The use a conformal transformation to map the problem onto an easier domain for which one can solve it.

ii) The use the Poisson integral formula.

In *i*) we have more the set up of a strategy rather than a concrete solution technique. With regard to the previous section we will study here mainly the cases *i*).

2.1 Potential theory (generalities)

In many cases the information about the function ψ in D is the Laplace equation

$$\Delta\psi = \partial_x^2\psi + \partial_y^2\psi = 0 \tag{2.1}$$

and one calls this kind of scenario *potential theory.* This means for this situation we will seek some harmonic function with given boundary condition, either of Neumann or Dirichlet type. If we want to follow the strategy outlined above and use conformal transformations, we have to ensure first that our original problem is not altered by such manipulations. Indeed this can be guaranteed by the following theorems.

Theorem 9: *A harmonic function $\psi(x, y)$ transforms into a harmonic function $\psi(u, v)$ when changing variables as $z = x+iy = f(w) = f(u+iv)$ with f being an analytic function.*

Proof : Take $\psi(x, y)$ to be a harmonic function and $z = f(w)$ to be an analytic function.

$\Rightarrow \exists$ a conjugate function $\tilde{\psi}(x, y)$ (see section 1.2.4)

$\Rightarrow$ The newly defined function $\phi(x,y) = \psi(x,y) + i\tilde{\psi}(x,y)$ is an analytic function of z, which follows by Corollary 1.

$\Rightarrow$ $\phi(z) = \phi(f(w))$ is an analytic function of w, since an analytic function of an analytic function is an analytic function.

$\Rightarrow$ ψ is a harmonic function of u, v.$\square$

Let us see what this means in a concrete example.
Example 1: In section 1.2.4. we have already encountered the following harmonic function

$$\psi(x,y) = x^2 - y^2 \quad \Rightarrow \quad \Delta_{xy}\psi = 0. \tag{2.2}$$

We write here Δ_{xy} to indicate that the derivatives are taken with respect to x, y. Now take as an analytic function $f(w) = e^{iw}$, which maps $w \mapsto z$

$$\begin{aligned} z &= f(w) = e^{iw} = e^{i(u+iv)} = e^{-v}e^{iu} \\ &= e^{-v}\cos u + ie^{-v}\sin u \\ &= x + iy. \end{aligned} \tag{2.3}$$

Comparing the real and imaginary part, we can read off from this $x(u,v) = e^{-v}\cos u$ and $y(u,v) = e^{-v}\sin u$ and use this to express the function $\psi(x,y)$ in (2.2) as a function in u and v

$$\psi(u,v) = e^{-2v}\cos^2 u - e^{-2v}\sin^2 u = e^{-2v}\cos 2u. \tag{2.4}$$

Next we verify that ψ is also a harmonic function in u, v

$$\left.\begin{aligned} \partial_u^2\psi(u,v) = \partial_u(-2\sin(2u)e^{-2v}) = -4e^{-2v}\cos(2u) \\ \partial_v^2\psi(u,v) = \partial_v(-2\cos(2u)e^{-2v}) = 4e^{-2v}\cos(2u) \end{aligned}\right\} \Rightarrow \Delta_{uv}\psi(u,v) = 0. \tag{2.5}$$

Having ensured that the harmonic nature of the function inside D does not change when we apply an analytic function to its argument, we have to see next what happens at the boundary.
Theorem 10: *Given a harmonic function $\psi(x,y)$ which is transformed by a conformal map $z = f(w)$ with $f'(w) \neq 0$. The boundary conditions which are either of the Dirichlet*

$$\psi(x,y) = \psi(x(u,v), y(u,v)) = c = \text{const}, \tag{2.6}$$

or Neumann type

$$\frac{d\psi}{dn_{x,y}} = \frac{d\psi}{dn_{u,v}} = 0 = \nabla_{x,y}\psi \cdot \vec{n}_{x,y} = \nabla_{u,v}\psi \cdot \vec{n}_{u,v}, \tag{2.7}$$

remain unchanged. Here $\vec{n}_{x,y}$ is a normal vector to the curve C_z parameterized by $\psi(x,y)$ in the z-plane and $\vec{n}_{u,v}$ is a normal vector to the curve C_w parameterized by $\psi(u,v)$ in the w-plane.

Proof : The conservation of the Dirichlet boundary conditions is obvious.

Next we consider the Neunmann type. Recall (from calculus 2) that the gradient vector

$$\nabla\psi(x,y) = \frac{\partial\psi(x,y)}{\partial x} + \frac{\partial\psi(x,y)}{\partial y} \tag{2.8}$$

at a particular point $z_0 = x_0 + iy_0$ of a function ψ points into the direction in which the directional derivative of ψ has its maximal value. See figure 15 for a schematic explanation. The maximum rate of change is $|\nabla\psi|$. It is clear that the gradient is orthogonal to the level curve through z_0 at which $\psi(x,y) = \text{const.}$ When we have a vanishing normal derivative $d\psi/dn_{x,y} = 0$ along a curve C_z in the z-plane, it means that the normal vector $\vec{n}_{x,y}$ is orthogonal to the gradient vector $\nabla_{x,y}\psi$. This is because $d\psi/dn_{x,y}$ is the projection of the gradient onto the normal vector. We can now carry out the same analysis at the image curve C_w in the w-plane. Since conformal transformations preserve angles between curves, it follows directly that we also have $\nabla_{u,v}\psi \cdot \vec{n}_{u,v} = 0.\square$

This means we can safely follow the strategy outlined at the beginning, which consists of first solving the boundary problem for some easy set up and then using some conformal transformations to treat more involved situations. Let us therefore start by solving two relatively easy boundary problems.

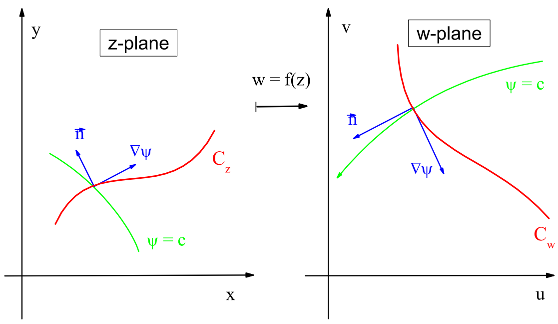


Figure 15: Neumann boundary condition.

2.2 Electrostatic potential between two infinite plates

Electrodynamics is governed by Maxwell's equations, which were formulated in 1864. In this course we do not want to concentrate on physics and therefore this will only be a sketchy introduction. In case you like to read up on the background: The classic book on the subject is "J.D. Jackson, Classical Electrodynamics, (Academic Press, New York, 1998, 3rd edition)".

In electrostatics, that means we do not have any time dependence, one of the Maxwell's equations reads

$$\nabla \cdot \vec{E} = 0, \tag{2.9}$$

where $\vec{E}$ is the electric field vector, which can be expressed as the negative of the gradient of a scalar potential ϕ

$$\vec{E} = -\nabla\phi. \tag{2.10}$$

When combining these two equations it follows directly that the scalar potential satisfies the Laplace equation

$$\nabla \cdot \nabla\phi = \Delta\phi = 0. \tag{2.11}$$

We shall now consider the problem of two infinite plates, which are located in the yz-plane at the positions $x = x_1$ and $x = x_1$, see figure 16. Clearly the geometry of this problem dictates that the potential depends only on the x-direction and must be independent of y and z. This means if you move along the y or z direction you do not notice any difference. The Laplace equation reduces therefore to

$$\Delta\phi = \frac{d^2\phi}{dx^2} = 0, \qquad (2.12)$$

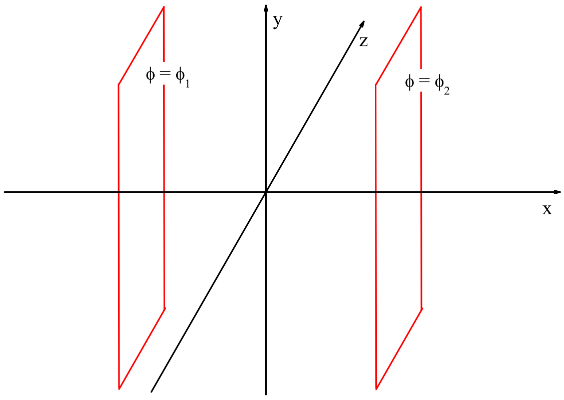


Figure 16: Infinite planes.

which is easily solved by

$$\phi(x) = \alpha x + \beta \qquad \alpha, \beta = \text{const}\,. \qquad (2.13)$$

Having solved the Laplace equation, we have to invoke the boundary conditions to fix the constants α, β. For simplicity we place now the plates at the position $x = \pm 1$. Therefore we obtain

$$\left.\begin{array}{l} \phi_1 = -\alpha + \beta \\ \phi_2 = \alpha + \beta \end{array}\right\} \Rightarrow \beta = \frac{1}{2}(\phi_1 + \phi_2), \alpha = \frac{1}{2}(\phi_2 - \phi_1) \qquad (2.14)$$

and finally the potential, which respects the given boundary conditions, is

$$\phi(x) = \frac{1}{2}(\phi_2 - \phi_1)x + \frac{1}{2}(\phi_1 + \phi_2). \qquad (2.15)$$

Having solved the Laplace equation in one dimension with given boundary conditions, let us look at a slightly more complicate situation in two dimension.

2.3 Electrostatic potential between two coaxial cylinders

We now want to find the potential function for two infinitely long coaxial cylinders with radii $r_0, r = 1$ at potentials ϕ_0, ϕ_1, respectively, see figure 17.

In order to incorporate the symmetry of the problem we consider the Laplace equation in polar coordinates as defined in (1.19) (recall calculus 2 or see exercise sheet 4)

$$\Delta\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0. \qquad (2.16)$$

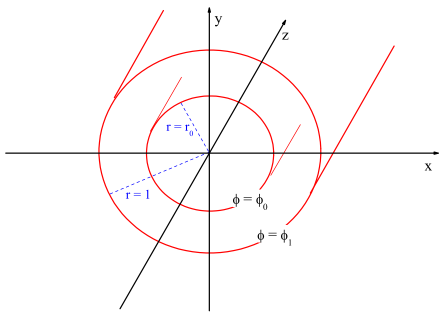


Figure 17: Coaxial cylinders.

According to the symmetry of the problem there can not be any ϑ dependence, such the $\partial\phi/\partial\vartheta = 0$. We may then write (2.16) as

$$\Delta\phi = 0 \qquad \Leftrightarrow \qquad r\frac{\partial^2\phi}{\partial r^2} + \frac{\partial\phi}{\partial r} = 0. \qquad (2.17)$$

Let us now solve this equation. Denoting first $\psi := \partial\phi/\partial r$ we can reduce the second order differential equation (2.17) to a first order differential equation

$$r\frac{\partial\psi}{\partial r} + \psi = 0. \tag{2.18}$$

Separation of variables then yields

$$\int \frac{1}{r} dr = -\int \frac{1}{\psi} d\psi \quad \Rightarrow \quad \ln r = -\ln\psi + c, \tag{2.19}$$

with c being some integration constant. Therefore we can write

$$\ln\psi = \ln\frac{\partial\phi}{\partial r} = -\ln r + c = \ln\frac{\kappa}{r}, \tag{2.20}$$

when introducing a new constant $c = \ln\kappa$. Exponentiating both sides and a subsequent integration yields

$$\frac{\partial\phi}{\partial r} = \frac{\kappa}{r} \quad \Rightarrow \quad \phi(r) = \kappa\ln r + \lambda, \tag{2.21}$$

with λ being a new integration constant. The constants κ, λ have to be determined from the boundary conditions, which when invoked give

$$\left.\begin{array}{l} \phi_1 = \kappa\ln(r=1) + \lambda = \lambda \\ \phi_0 = \kappa\ln r_0 + \lambda \end{array}\right\} \Rightarrow \kappa = \frac{\phi_0 - \phi_1}{\ln r_0}. \tag{2.22}$$

Finally the potential which respects the given boundary conditions is

$$\phi(r) = (\phi_0 - \phi_1)\frac{\ln r}{\ln r_0} + \phi_1. \tag{2.23}$$

Having solved the Laplace equation in two cases directly we proceed to consider a less symmetrical geometric configuration and follow the strategy outlined above, namely using a conformal transformation to map it to one of the previous problems.

2.4 Electrostatic potential between two non-coaxial cylinders

We shall now try to find the potential function for the entire z-plane when the two infinitely long cylinders $|z| = 1$ and $|z - x_0| = x_0$ are non-coaxial. We understand here that the z-plane is the xy-plane, which is not to be confused with the z-direction, i.e. by $|z| = 1$ we mean $x^2 + y^2 = 1$, etc. We place the cylinders at the constant potentials $\phi_1 = 0$ at $|z| = 1$ and $\phi_0 = 220V$ at $|z - x_0| = x_0$. We will take the value of the center of the smaller cylinder and its radius to be $x_0 = 3/10$ when it is convenient.

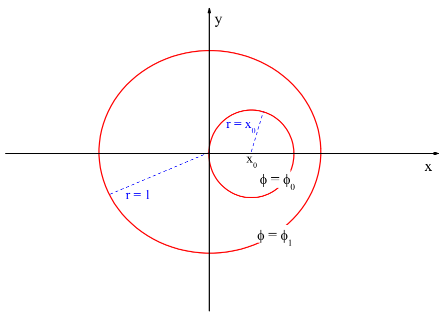


Figure 18: Non coaxial cylinders.

Solution : In order to make use of the result we found in the previous section, we have to find a map which leaves the circle with the radius $|z| = 1$ invariant, that is we want the image to be $|w| = 1$. In addition we want to map the circle $|z - x_0| = x_0$ to $|w| = r_0$. Let as make an inspired guess and assume this map is of the form of a linear fractional transformation (1.77). We have seen that we can fix this map by means of three points. Let us map the three points $z_1 = 1, z_2 = -1, z_3 = i$ on the unit circle onto the three points on the unit circle in the image plane

$$w_1 = -i, \qquad w_2 = 1, \qquad w_3 = \frac{2c}{1+c^2} + i\frac{(c^2-1)}{1+c^2}, \quad c \in \mathbb{R}. \tag{2.24}$$

It is slightly less obvious that w_3 is on the unit circle, but one can easily check that indeed $(\mathrm{Re}\, w_3)^2 + (\mathrm{Im}\, w_3)^2 = 1$. This choice seems complicated at this stage, but we have achieved that there is a free parameter in our set of equations. Substituting these points into (1.86) and solving for w in the usual way we find[2]

$$w = f(z) = \frac{z-c}{cz-1} \qquad \text{with } c \in \mathbb{R}. \tag{2.25}$$

We can now exploit the fact that c is a free parameter. Our geometric set up also dictates that

$$f(0) = r_0 \qquad \Rightarrow \qquad c = r_0 \tag{2.26}$$

$$f(2x_0) = -r_0 \qquad \Rightarrow \qquad \frac{2x_0 - r_0}{2x_0 r_0 - 1} = -r_0 \tag{2.27}$$

Combining these equations yields a quadratic equation in r_0

$$2x_0 - r_0 + 2x_0 r_0^2 - r_0 = 0 \quad \Rightarrow \quad r_0^{(1/2)} = \frac{1}{2x_0} \pm \frac{1}{2x_0}\sqrt{1-4x_0^2}. \tag{2.28}$$

Taking now $x_0 = 3/10$ gives $r_0^{(1)} = 1/3$ and $r_0^{(2)} = 3$. Since we want the cylinder at radius r_0 to be in the inside of the cylinder with radius $r = 1$, i.e. $r_0 < 1$, we discard the solution $r_0^{(2)}$. Therefore we find that the resulting conformal map is

$$f(z) = \frac{z - 1/3}{z/3 - 1} = \frac{3z-1}{z-3}. \tag{2.29}$$

We can now employ the solution of the previous section and compute the potential for two non-coaxial infinite cylinders to

$$\phi(r) = (\phi_0 - \phi_1)\frac{\ln(f(r))}{\ln r_0} + \phi_1 = -220\frac{\ln\left|\frac{3z-1}{z-3}\right|}{\ln 3} V. \tag{2.30}$$

[2]Note that this is a special case of the most general linear fractional transformation that maps a circle of radius one into a circle of radius one

$$T(z) = e^{i\theta}\frac{z-\gamma}{\bar{\gamma}z - 1} \qquad \text{for } \theta \in \mathbb{R}, \gamma \in \mathbb{C},$$

as proven in the coursework.

2.5 Thermal conductivity, steady state temperature between infinite plates

Under steady state conditions, which means we have no time dependence, the balance of the heat flow through a solid yields the Laplace equation for the temperature T

$$\Delta T = \partial_x^2 T + \partial_y^2 T = 0. \tag{2.31}$$

Now we wish to determine the temperature between two walls at positions $x = \pm\pi/2$, which are kept at temperature $T = 0$. The walls stand on a surface which is kept at temperature $T = 1$. See figure 19 for the geometrical set up. This means we have to solve the following Dirichlet problem

$$\Delta T = 0, \quad T(\pm\pi/2, y) = 0, \quad T(x, 0) = 1, \quad \text{for } |x| < \frac{\pi}{2}, y > 0. \tag{2.32}$$

This is difficult to solve directly. Suppose instead that we had to solve the easier problem of two parallel infinite plates

$$\Delta T = 0, \quad T(x, 0) = 0, \quad T(x, \pi) = 1, \quad \text{for } 0 < y < \pi. \tag{2.33}$$

This is solved directly by

$$T(x, y) = \frac{y}{\pi}, \tag{2.34}$$

as one can check. As in the previous section we try to make use of this solution and map the problem (2.32) onto (2.33). Indeed we can employ the maps

$$\begin{aligned} \tilde{f} &: z \mapsto \tilde{w} & \tilde{f}(z) &= \sin z, \\ \hat{f} &: \tilde{w} \mapsto w & \hat{f}(\tilde{w}) &= \ln\left(\frac{\tilde{w}-1}{\tilde{w}+1}\right), \\ f &: z \mapsto w & f(z) &= \hat{f} \circ \tilde{f}(z). \end{aligned}$$

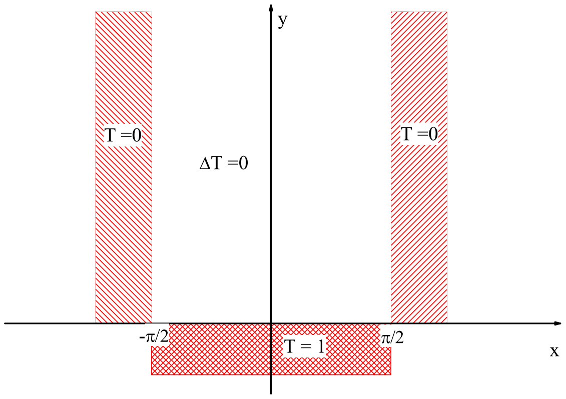


Figure 19: Temperature in semi-infinite walls.

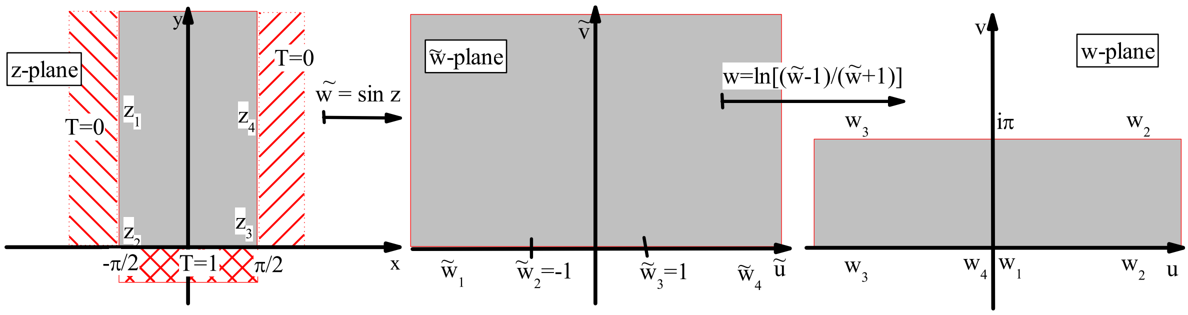


Figure 20: Conformal mapping of the semi-infinite strip to an infinite strip.

Here the maps are simply given. In order to verify that they map indeed as indicated in figure 20, let us parameterize at first the boundaries and investigate how they are transformed. We use the identity

$$\begin{aligned}\tilde{f}(z) &= \sin z = \sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy) \\ &= \sin x \cosh y + i \cos x \sinh y.\end{aligned} \tag{2.35}$$

Then the vertival line $-\pi/2 + iy$ with $y > 0$ is mapped as follows

$$\tilde{f}\left(-\frac{\pi}{2} + iy\right) = \sin\left(-\frac{\pi}{2}\right)\cosh y + i\cos\left(-\frac{\pi}{2}\right)\sinh y = -\cosh y. \tag{2.36}$$

Since $-\infty < \cosh y < 1$ we deduce that the vertical line $-\pi/2 + iy$ is mapped to the horizontal line $x < -1$ by $\tilde{f}$ as indicated in figure 20. Considering next the vertival line $\pi/2 + iy$ with $y > 0$ we find

$$\tilde{f}\left(\frac{\pi}{2} + iy\right) = \sin\left(\frac{\pi}{2}\right)\cosh y + i\cos\left(\frac{\pi}{2}\right)\sinh y = \cosh y, \tag{2.37}$$

such that we deduce that this vertical line is mapped to the horizontal line $x > 1$ by $\tilde{f}$ as indicated in figure 20. Finally we also map the horizontal line segment $-\pi/2 < x < \pi/2$

$$\tilde{f}\,(x) = \sin{(x)}\cosh 0 + i\cos{(x)}\sinh 0 = \sin x, \tag{2.38}$$

which produces the horizontal line segment $-1 < x < 1$. Having established that the boundaries are mapped as indicated we still have to investigate what happens to the interior. Since the region is connected it is sufficient to investigate the map of one point. For instance we compute $\tilde{f}\,(i) = i \sinh 1$ which is in the upper half plane. We conclude from this that indeed $\tilde{f}(z)$ maps the region $-\pi/2 < x < \pi/2$, $y \geq 0$ to the upper half plane. Similarly we can show that $\hat{f}$ maps the upper half plane to the vertical strip $-\infty < x < \infty$, $0 < y < \pi$.

Therefore, the solution to the first Dirichlet problem (2.32) is

$$T(x,y) = \frac{1}{\pi}\,\mathrm{Im}(\hat{f} \circ \tilde{f}(z)) = \frac{1}{\pi}\,\mathrm{Im}\left[\ln\left(\frac{\sin z - 1}{\sin z + 1}\right)\right]. \tag{2.39}$$

Let us work this out in more detail and simplify this. First we compute, recall (1.20) for this

$$w = \ln\left(\frac{\tilde{w}-1}{\tilde{w}+1}\right) = \ln\left|\frac{\tilde{w}-1}{\tilde{w}+1}\right| + i\arg\left(\frac{\tilde{w}-1}{\tilde{w}+1}\right) = u + iv, \tag{2.40}$$

such that

$$\begin{aligned}\mathrm{Im}\, w &= \arg\left(\frac{\tilde{w}-1}{\tilde{w}+1}\right) = \arg\left(\frac{\tilde{x}+i\tilde{y}-1}{\tilde{x}+i\tilde{y}+1}\right) = \arg\left[\frac{(\tilde{x}+i\tilde{y}-1)(\tilde{x}+1-i\tilde{y})}{(\tilde{x}+1+i\tilde{y})(\tilde{x}+1-i\tilde{y})}\right], \\ &= \arg\left(\frac{\tilde{x}^2+\tilde{y}^2-1+i2\tilde{y})}{(\tilde{x}+1)^2+\tilde{y}^2}\right) = \arctan\left(\frac{2\tilde{y}}{\tilde{x}^2+\tilde{y}^2-1}\right).\end{aligned} \tag{2.41}$$

In the last equality we used (1.21). Next we need to express $\tilde{x}, \tilde{y}$ in terms of x, y. We have

$$\tilde{w} = \sin(z) = \sin(x+iy) = \sin x \cosh y + i\cos x \sinh y = \tilde{x} + i\tilde{y} \tag{2.42}$$

and therefore

$$\begin{aligned}
\operatorname{Im} w &= \arctan\left(\frac{2\cos x \sinh y}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y - 1}\right) \qquad (2.43)\\
&= \arctan\left(\frac{2\cos x \sinh y}{\sinh^2 y - \cos^2 x}\right) = \arctan\left(\frac{2\cos x/\sinh y}{1-\cos^2 x/\sinh^2 y}\right).
\end{aligned}$$

Introducing now the auxiliary variable $\tan\gamma = \cos x/\sinh y$ we can use the identity $\tan 2\gamma = 2\tan\gamma/(1-\tan^2\gamma)$ and obtain

$$\operatorname{Im} w = \arctan(\tan 2\gamma) = 2\gamma = 2\arctan\left(\frac{\cos x}{\sinh y}\right). \qquad (2.44)$$

This means the solution to the boundary Dirichlet problem (2.32) is

$$T(x,y) = \frac{2}{\pi}\arctan\left(\frac{\cos x}{\sinh y}\right). \qquad (2.45)$$

We may easily check that the boundary condition are indeed satisfied

$$\begin{aligned}
T(\pm\pi/2, y) &= \frac{2}{\pi}\arctan\left(\frac{\cos(\pm\pi/2)}{\sinh y}\right) = \frac{2}{\pi}\arctan 0 = 0, \qquad (2.46)\\
T(x,0) &= \frac{2}{\pi}\arctan\left(\frac{\cos x}{\sinh 0}\right) = \frac{2}{\pi}\lim_{x\to\infty}\arctan(x) = 1. \qquad (2.47)
\end{aligned}$$

More work is to check that (2.45) satisfies also the Laplace equation. We leave this as an exercise.

3. Transform Methods

In this part of the course we will study various transform methods, which also allow to solve boundary value problems of certain type of differential equations. An integral transform is a relation of the form

$$F(x) = \int_\alpha^\beta K(x,t) f(t) dt. \qquad (3.1)$$

The function F is said to be the *transform* of f and K is called the *kernel* of the transformation.

3.1 The Fourier Transform

First recall that a periodic function $f(x) = f(x+2\pi)$ can be expanded in terms of exponential functions

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}, \qquad (3.2)$$

where the c_k are constants. Often more familiar is an expansion in terms of trigonometric functions, but this can be achieved simply by using Euler's identity for $e^{ikx} = \cos x + i\sin x$. Such a series is called a *Fourier series*. Terminating the sum at some finite value gives often a very good approximation for $f(x)$. When $f(x)$ is not periodic such an expansion

is no longer valid, but one may replace the sum in (3.2) by an integral and then obtains a meaningful expression under certain circumstances. Let us define this more precisely:
Definition: *The Fourier transform* $\mathcal{F}$u(x)=$\hat{u}$*(x) of a piecewise smooth and absolutely integrable function u(x) on the real line is defined as*

$$\mathcal{F}u(x) := \hat{u}(x) = \int_{-\infty}^{\infty} u(t)e^{-itx}dt. \tag{3.3}$$

We have quoted here various notations which can be found in the literature $\mathcal{F}u(x) = Fu(x) = \hat{u}(x) = \tilde{u}(x) \ldots$ All of them are defined by the right hand side of (3.3). Here we will mostly use $\mathcal{F}u(x)$ and sometimes $\hat{u}(x)$. In the definition we have employed a few notions, which we need to specify in more detail.
Definition: *A function u(x) is said to be absolutely integrable when*

$$\int_{-\infty}^{\infty} |u(t)|\, dt < \infty. \tag{3.4}$$

Definition: *A function u(x) is said to be piecewise smooth when there exist a finite number of points $x_1 < x_2 < \ldots < x_n$ on the real axis such that*

i) u(x) is continuous on all the intervals (-∞, x_1), (x_1, x_2),... (x_n, ∞).

ii) the left and right limits of u(x) exists on all points $x_1, x_2, \ldots, x_n$.

Note that we do not demand that the left and right limits of $u(x)$ coincide at the points $x_1, x_2, \ldots, x_n$. Let us see how to compute $\mathcal{F}u(x)$.
Example 1: Compute the Fourier transforms of the function

$$u(x) = \begin{cases} 1 & \text{for } |x| < \lambda \\ 0 & \text{for } |x| > \lambda \end{cases}. \tag{3.5}$$

Solution : First we verify that $u(x)$ is piecewise smooth. Except at $x = \pm\lambda$ the function is continuous, such that only at these two points we might encounter a problem. The left and right limits at these points exist. The left limits are

$$\lim_{\varepsilon\to 0} u(\lambda - \varepsilon) = 1 \qquad \lim_{\varepsilon\to 0} u(-\lambda - \varepsilon) = 0 \tag{3.6}$$

and the right limits are

$$\lim_{\varepsilon\to 0} u(\lambda + \varepsilon) = 0 \qquad \lim_{\varepsilon\to 0} u(-\lambda + \varepsilon) = 1. \tag{3.7}$$

The function $u(x)$ is also absolutely integrable

$$\int_{-\infty}^{\infty} |u(t)|\, dt = \int_{-\lambda}^{\lambda} 1dt = 2\lambda < \infty. \tag{3.8}$$

From the definition of the Fourier transform (3.3) follows

$$\mathcal{F}u(x) = \int_{-\lambda}^{\lambda} e^{-itx}dt = \frac{i}{x}e^{-itx}\Big|_{-\lambda}^{\lambda} = 2\frac{\sin\lambda x}{x}. \tag{3.9}$$

Example 2: Compute the Fourier transform of the function

$$u(x) = \begin{cases} \frac{1}{x} & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases} . \tag{3.10}$$

Solution : In this case the left limit at $x = 0$ is not finite, such that $u(x)$ is not piecewise smooth and therefore the Fourier transform does not exit.

Example 3: Compute the Fourier transform of the function

$$u(x) = \begin{cases} \ln x & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{3.11}$$

Solution : In this case the right limit at $x = 0$ is not finite, such that $u(x)$ is not piecewise smooth and therefore the Fourier transform does not exit.

Example 4: Compute the Fourier transform of the *Heavyside function* (unit step function)

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases} . \tag{3.12}$$

Solution : In this case the function is not absolutely integrable since $\int_{-\infty}^{\infty} |u(t)|\, dt = \int_0^\infty dt \to \infty$. Therefore the Fourier transform of $u(x)$ does not exist. We will see later that the Laplace transform is a more suitable integral transform to handle this function.

Example 5: Compute the Fourier transform of the function

$$u(x) = e^{-x^2} \tag{3.13}$$

Solution : Clearly $u(x)$ is piecewise smooth. From the definition of the Fourier transform (3.3) follows

$$\mathcal{F}u(x) = \int_{-\infty}^{\infty} e^{-t^2} e^{-itx} dt = \int_{-\infty}^{\infty} e^{-(t+ix/2)^2} e^{-x^2/4} dt \tag{3.14}$$

$$= e^{-x^2/4} \int_{-\infty}^{\infty} e^{-(t+ix/2)^2} dt = \sqrt{\pi} e^{-x^2/4}. \tag{3.15}$$

We assume here that we know the integral $\int_{-\infty}^{\infty} e^{-(t+ix/2)^2} dt = \sqrt{\pi}$, which can be computed by integrating in the complex plane.

3.1.1 Properties of the Fourier transform

Before looking at some application for Fourier transforms, let us first study some of its basic properties:

i) Linearity: The Fourier transform acts linearly

$$\mathcal{F}(\lambda u + \kappa v)(x) = \lambda \mathcal{F}(u)(x) + \kappa \mathcal{F}(v)(x), \qquad \lambda, \kappa \in \mathbb{C} \tag{3.16}$$

which follows trivially from the definition (3.3).

ii) Translation: The Fourier transform for the shifted function $v(x) = u(x+\Delta)$ with $\Delta \in \mathbb{R}$ is

$$\mathcal{F}v(x) = \int_{-\infty}^{\infty} u(t+\Delta) e^{-itx} dt \underset{t \to t-\Delta}{=} e^{ix\Delta} \mathcal{F}u(x). \tag{3.17}$$

iii) Scaling: The Fourier transform for the scaled function $v(x) = u(\lambda x)$ with $\lambda \in \mathbb{R}^+$ is

$$\mathcal{F}v(x) = \mathcal{F}u(\lambda x) = \int_{-\infty}^{\infty} u(\lambda t) e^{-itx} dt \underset{t \to t/\lambda}{=} \frac{1}{\lambda} \int_{-\infty}^{\infty} u(t) e^{-itx/\lambda} dt = \frac{1}{\lambda} \hat{u}(x/\lambda). \tag{3.18}$$

Note here that $\hat{u}(x/\lambda)$ means $(\mathcal{F}u)(x/\lambda)$ and not $\mathcal{F}u(x/\lambda)$.

iv) Derivative of u: The Fourier transform for the derivative $u'(x)$ of the function $u(x)$ is

$$\mathcal{F}u'(x) = \int_{-\infty}^{\infty} u'(t) e^{-itx} dt = u(t) e^{-itx}|_{-\infty}^{\infty} + ix \int_{-\infty}^{\infty} u(t) e^{-itx} dt = ix\mathcal{F}u(x). \tag{3.19}$$

when $\lim_{t \to \pm\infty} u(t) = 0$. We integrated here by parts. Note this is of course not to be confused with the derivative of $\mathcal{F}u(x)$.

3.1.2 The convolution of two functions

Definition: *The convolution of two functions u(x) and v(x) is defined as*

$$u \star v(x) = \int_{-\infty}^{\infty} u(t) v(x-t) dt. \tag{3.20}$$

The convolution satisfies an important property, which we will exploit below.
Lemma 3: *The Fourier transform of the convolution $u \star v(x)$ equals the product of the Fourier transforms of u and v*

$$\mathcal{F}(u \star v)(x) = (\mathcal{F}u)(x)(\mathcal{F}v)(x). \tag{3.21}$$

Proof : The proof is straightforward. From the definition (3.3) follows

$$\begin{aligned}
\mathcal{F}(u \star v)(x) &= \int_{-\infty}^{\infty} (u \star v)(t) e^{-itx} dt = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds\ u(s) v(t-s) e^{-itx} \\
&= \int_{-\infty}^{\infty} ds\ u(s) \left(\int_{-\infty}^{\infty} dt\ v(t-s) e^{-itx} e^{isx} \right) e^{-isx} \\
&= \int_{-\infty}^{\infty} ds\ u(s) e^{-isx} \left(\int_{-\infty}^{\infty} dt\ v(t) e^{-itx} \right) \\
&= \mathcal{F}(u)\mathcal{F}(v).
\end{aligned}$$

We are now equipped to solve a concrete boundary value problem by means of Fourier transforms.□

3.1.3 An application of the Fourier transform, the heat equation

We consider a one-dimensional infinite rod. The temperature $T(t,x)$ is a function of the time t and position x, which has to satisfy the *heat equation*

$$\frac{\partial T(t,x)}{\partial t} = \frac{\partial^2 T(t,x)}{\partial x^2}. \tag{3.22}$$

Notice that under steady state conditions the equation reduces to the Laplace equation in one dimension, which we have encountered before. Now the new feature of a time dependence enters into our considerations. As initial condition we take

$$T(0,x) = f(x) \qquad \text{for } -\infty < x < \infty, \tag{3.23}$$

with some function $f(x)$ which we do not specify further at this stage. We assume that the temperature at the end of the rod is zero and that the derivatives vanish. This means our boundary conditions are of Dirichlet and Neumann type

$$\lim_{x\to\pm\infty} T(t,x) = \lim_{x\to\pm\infty} \partial_x T(t,x) = 0. \tag{3.24}$$

We will now employ the Fourier transform in the space variable x to solve the heat equation. From the definition of the Fourier transform (3.3) of $T(t,x)$ we have

$$\mathcal{F}_x T(t,x)\hat{T}(t,x) = \int_{-\infty}^{\infty} T(t,s)e^{-isx}ds. \tag{3.25}$$

Since T depends now on two variables x and t we have indicated here explicitly in which variable we take the Fourier transform, i.e. in x. Taking the derivative of this equation and using subsequently the heat equation gives

$$\partial_t \hat{T}(t,x) = \int_{-\infty}^{\infty} \partial_t T(t,s)e^{-isx}ds = \int_{-\infty}^{\infty} \partial_s^2 T(t,s)e^{-isx}ds. \tag{3.26}$$

Integrating now twice by parts and using the boundary conditions (3.24) gives

$$\begin{aligned}
\partial_t \hat{T}(t,x) &= \partial_s T(t,s)e^{-isx}\big|_{-\infty}^{\infty} + ix\int_{-\infty}^{\infty} \partial_s T(t,s)e^{-isx}ds \\
&= \partial_s T(t,s)e^{-isx}\big|_{-\infty}^{\infty} + ix\, T(t,s)e^{-isx}\big|_{-\infty}^{\infty} - x^2\int_{-\infty}^{\infty} T(t,s)e^{-isx}ds \\
&= -x^2\int_{-\infty}^{\infty} T(t,s)e^{-isx}ds = -x^2\hat{T}(t,x).
\end{aligned} \tag{3.27}$$

Next we invoke the initial condition (3.23)

$$\hat{T}(0,x) = \int_{-\infty}^{\infty} T(0,s)e^{-isx}ds = \int_{-\infty}^{\infty} f(s)e^{-isx}ds = \hat{f}(x). \tag{3.28}$$

Thus we have reduces the original problem of a second order differential equation (3.22) with given boundary condition (3.24) to a first order differential equation with given boundary condition

$$\partial_t \hat{T}(t,x) = -x^2\hat{T}(t,x)\ , \qquad \hat{T}(0,x) = \hat{f}(x). \tag{3.29}$$

The equation (3.29) is solved by

$$\hat{T}(t,x) = \hat{f}(x)e^{-tx^2}. \tag{3.30}$$

From our example 5 we know that $\mathcal{F}u(x) = \sqrt{\pi}e^{-x^2/4} = \hat{u}(x)$ for $u(x) = e^{-x^2}$. Recalling also the scaling property (3.18) we deduce that

$$\mathcal{F}(e^{-x^2/4t}) = 2\sqrt{t}\hat{u}(2\sqrt{t}x) = 2\sqrt{\pi t}e^{-tx^2} \qquad \text{for } x \in \mathbb{R}. \tag{3.31}$$

This means we can deduce that e^{-tx^2} is the Fourier transform of the function

$$K(x) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}, \tag{3.32}$$

such that (3.30) maybe rewritten as

$$\mathcal{F}T(t,x) = \hat{f}(x)\hat{K}(x) = \mathcal{F}(f \star K)(x), \tag{3.33}$$

where we used the convolution theorem (3.20) in the last equality. We may now drop the $\mathcal{F}$ from (3.33) or more formally act with the inverse Fourier transform on both sides and express the temperature function $T(t,x)$ as a convolution between the functions f and K

$$T(t,x) = \int_{-\infty}^{\infty} f(y)\frac{1}{2\sqrt{\pi t}}e^{-i(x-y)^2/4t}dy = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(y)e^{-i(x-y)^2/4t}dy. \tag{3.34}$$

This solves the heat equation with given boundary conditions and once the initial condition is also given we are only left with one integral.

3.2 The Laplace transform

The Laplace transform of a function $u(x)$ is essentially the Fourier transform of this function whose argument is rotated by $-\pi/2$

$$\mathcal{F}u(f_R^{-\pi/2}x) = \mathcal{F}u(e^{-i\pi/2}x) = \mathcal{F}u(-ix) = \int_{-\infty}^{\infty} u(t)e^{-xt}dt. \tag{3.35}$$

Since in comparison with the Fourier transform we have now traded the oscillatory function e^{-ixt} for e^{-xt}, we require a very strong decay for $u(t)$ when $t \to -\infty$. To avoid this one usually assumes that $u(x) = 0$ for $-\infty < x < 0$. This does not really pose a serious restriction as in many ways this makes the Laplace transform actually more physical (more realistic) than the Fourier transform. Usually in a physical context something, e.g. a laser pulse, an electric or magnetic signal, etc., is switched on at some point in time. Without loss of generality we may set this point in time to $t = 0$. Only from that moment onwards one would like to study the system and has no interest in what happened before. Let us assemble these notions and define the Laplace transform more formally:
Definition: *The* Laplace transform $\mathcal{L}$u(x) *of a piecewise smooth function* u(x) *with exponential growth α is defined as*

$$\mathcal{L}u(x) := \int_0^{\infty} u(t)e^{-tx}dt \qquad \text{for } x > \alpha. \tag{3.36}$$

We need to specify what we mean by exponential growth and explain why we require this new notion here.

Definition: *The function* u(x) *is said to have* exponential growth α *if there exists a constant* μ *such that*

$$|u(x)| \leq \mu e^{\alpha x} \qquad \text{for } x > 0, \text{ with } \alpha, \mu \in \mathbb{R}. \tag{3.37}$$

Notice that the important role is played here by the constant α as this leads to a restriction for x in (3.36) for the validity of the Laplace transform. The constant μ only has to exist and makes no further impact. Let us see why this is a useful property to have. The knowledge of how the function grows can be used to ensure the existence of the integral, as one can see by the following argument

$$\mathcal{L}u(x) \leq \int_0^\infty |u(t)|\, e^{-tx} dt \leq \mu \int_0^\infty e^{\alpha x} e^{-tx} dt = \mu \int_0^\infty e^{(\alpha - x)t} dt = \frac{\mu}{x - \alpha} < \infty \quad \text{for } x > \alpha. \tag{3.38}$$

The first inequality simply follows from the obvious fact that $\int f(t)dt \leq \int |f(t)|\, dt$. In the second inequality we used the definition of the exponential growth (3.37) and the rest is a straightforward computation of the integral.

Let us compute some explicit examples of Laplace transforms[3]:

Example 1: Compute the Laplace transform of the Heavyside function

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases} . \tag{3.41}$$

Solution : We have seen in example 4 of the previous section that we can not compute the Fourier transform of this function. Computing now the Laplace transform there is no problem with the convergence of the integral. Obviously $H(x)$ is piecewise smooth. Furthermore, the function is of exponential growth $\alpha = 0$, since

$$|H(x)| = 1 \leq \mu \quad \text{ for } \mu \geq 1. \tag{3.42}$$

We then compute

$$\mathcal{L}H(x) = \int_0^\infty e^{-tx} dt = -\frac{1}{x} e^{-tx} \Big|_0^\infty = \frac{1}{x} \qquad \text{for } x > \alpha = 0. \tag{3.43}$$

[3] **!!!** Warning: The notation for the Laplace transform can be confusing at times in the literature. One finds usually

$$u(x) \qquad \Leftrightarrow \qquad (\mathcal{L}u)(s) \; . \tag{3.39}$$

When we do not express u as a function of x in $\mathcal{L}u$ there is no confusion and one could also write $(\mathcal{L}u)(x)$ instead of $(\mathcal{L}u)(s)$. However, when we have a concrete functions, say $u(x) = e^{\alpha x}$, one often finds

$$u(x) = e^{\alpha x} \qquad \Leftrightarrow \qquad (\mathcal{L}e^{\alpha x})(s) = \frac{1}{s - \alpha} \quad \text{or} \quad (\mathcal{L}e^{\alpha s})(x) = \frac{1}{x - \alpha}. \tag{3.40}$$

Here one needs to be very clear about the roles played by the parameters s and x, as one has the rather odd situation of variable appearing on the left hand side of the equation which are absent on the right hand side. We avoid this here usually by writing the function name instead of its explicit form.

Note that the restriction we found from the exponential growth for x automatically takes care of the singular point $x = 0$, which would otherwise be ill defined in our final answer.

Example 2: Compute the Laplace transform of the function

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases} . \tag{3.44}$$

Solution : The function $u(x)$ is piecewise smooth and of exponential growth $\alpha = \varepsilon > 0$, since

$$|u(x)| = x \leq \mu e^{\varepsilon x} \qquad \text{for } \mu = x,\ \varepsilon > 0. \tag{3.45}$$

Integrating by parts[4] (with $u = t, v' = e^{-tx}, u' = 1, v = -1/xe^{-xt}$) gives

$$\mathcal{L}u(x) = \int_0^\infty te^{-tx}dt = -\frac{t}{x}e^{-tx}\Big|_0^\infty + \frac{1}{x}\int_0^\infty e^{-tx}dt = \frac{1}{x^2} \quad \text{for } x > \alpha = \varepsilon = 0. \tag{3.46}$$

We can generalize this to $u(x) = x^k$ by integrating k times by parts. Then we find (exercise)

$$\mathcal{L}u(x) = \frac{k!}{x^{k+1}} \quad \text{for } x > \alpha = \varepsilon = 0. \tag{3.47}$$

Obviously (3.47) reduces to (3.46) for $k = 1$.

Example 3: Compute the Laplace transform of the function

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{\beta x} & \text{for } x \geq 0 \end{cases} . \tag{3.48}$$

with $\beta \in \mathbb{R}$.

Solution : The function $u(x)$ is piecewise smooth and of exponential growth $\alpha = \beta$, since

$$|u(x)| = e^{\beta x} \leq \mu e^{\beta x} \qquad \text{for } \mu \geq 1. \tag{3.49}$$

We compute

$$\mathcal{L}u(x) = \int_0^\infty e^{\beta t}e^{-tx}dt = \int_0^\infty e^{(\beta-x)t}dt = \frac{e^{(\beta-x)t}}{\beta - x}\Bigg|_0^\infty = \frac{1}{x-\beta} \quad \text{for } \ x > \beta. \tag{3.50}$$

Example 4: Compute the Laplace transform of the function

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ \sin \lambda x & \text{for } x \geq 0 \end{cases} . \tag{3.51}$$

with $\lambda \in \mathbb{R}$.

[4]Recall the conventions $\int_\alpha^\beta uv'dx = uv|_\alpha^\beta - \int_\alpha^\beta u'vdx$.

Solution : The function $u(x)$ is piecewise smooth and of exponential growth $\alpha = 0$, since

$$|u(x)| = \sin \lambda x \leq \mu \quad \text{for } \mu \geq 1. \tag{3.52}$$

Integrating by parts ($u = e^{-tx}, v' = \sin \lambda t, u' = -xe^{-xt}, v = -1/\lambda \cos \lambda t$) gives

$$\begin{aligned} \mathcal{L}u(x) &= \int_0^\infty \sin \lambda t e^{-tx} dt = -\frac{1}{\lambda} \cos(\lambda t) e^{-tx} \Big|_0^\infty - \frac{x}{\lambda} \int_0^\infty \cos(\lambda t) e^{-tx} dt \\ &= \frac{1}{\lambda} - \frac{x}{\lambda} \int_0^\infty \cos(\lambda t) e^{-tx} dt \end{aligned} \tag{3.53}$$

Integrating once more by parts (with $u = e^{-tx}, v' = \cos \lambda t, u' = -xe^{-xt}, v = 1/\lambda \sin \lambda t$) gives

$$\begin{aligned} \mathcal{L}u(x) &= \frac{1}{\lambda} - \frac{x}{\lambda} \left[\frac{1}{\lambda} \sin(\lambda t) e^{-tx} \Big|_0^\infty + \frac{x}{\lambda} \int_0^\infty \sin(\lambda t) e^{-tx} dt \right] \\ &= \frac{1}{\lambda} - \frac{x^2}{\lambda^2} \mathcal{L}u(x). \end{aligned} \tag{3.54}$$

We can solve this for $\mathcal{L}u(x)$ and find

$$\mathcal{L}u(x) = \frac{\lambda}{\lambda^2 + x^2} \qquad \text{for } x > \alpha = 0. \tag{3.55}$$

Example 5: Compute once more the Laplace transform of the Heavyside function, but with shifted argument $v(x) = H(x - \alpha)$ where $\alpha \in \mathbb{R}^+$.

Solution : $H(x)$ is of course still piecewise smooth and of exponential growth $\alpha = 0$. Now we compute

$$\mathcal{L}v(x) = \int_0^\infty H(t - \alpha) e^{-tx} dt = \int_\alpha^\infty e^{-tx} dt = -\frac{1}{x} e^{-tx} \Big|_\alpha^\infty = \frac{1}{x} e^{-\alpha x}. \tag{3.56}$$

Alternatively we can integrate by parts (with $u = H(t - \alpha), v' = e^{-tx}, u' = H'(t - \alpha), v = -1/xe^{-xt}$). We do not know what $H'(t - \alpha)$ means and in fact we should be quite worried because $H(t - \alpha)$ is not continuous at $t = \alpha$, i.e. it is not well defined at all. Despite these concerns we compute

$$\mathcal{L}u(x) = -\frac{1}{x} H(t - \alpha) e^{-tx} \Big|_0^\infty + \int_0^\infty H'(t - \alpha) \frac{1}{x} e^{-tx} dt \tag{3.57}$$

$$= \int_0^\infty H'(t - \alpha) \frac{1}{x} e^{-tx} dt = \frac{1}{x} e^{-\alpha x}. \tag{3.58}$$

In the last equality we have used the result from (3.56). We do not know what the derivative $H'(t-\alpha)$ means by itself, but the relation (3.58) tells us about its behaviour under an integral. It has the effect that the other function under the integral just acquires the value at the discontinuity. We take this observation as a motivation for defining a new function:

Definition: *The* <u>*Dirac delta "function"*</u> $\delta(x-\alpha) := H'(x-\alpha)$ *is defined through the property*

$$f(\alpha) = \int_{-\infty}^{\infty} \delta(x-\alpha) f(x) dx. \tag{3.59}$$

The quotation marks indicate that despite the fact that in the literature $\delta(x)$ is referred to as a function it is not. Strictly speaking the expression $\delta(x)$ is a distribution and only makes proper sense under an integral. Intuitively we can understand this behaviour $H'(x-\alpha)$ is zero everywhere, except at $x = \alpha$, where it peaks very strongly. Therefore the function $f(x)$ under the integral is multiplied always by zero except at $x = \alpha$. We will make use of this function below.

We will not discuss the Dirac delta function in detail, but we will exploit below one of its properties. It admits the integral representation

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} dt. \tag{3.60}$$

We will not derive this representation in detail but simply try to understand it intuitively. The integral should yield 0 everywhere except for $x = 0$. Thinking of e^{itx} as $\cos x + i \sin x$ it is clear we sum up equal positive and negative areas, such that overall we obtain 0. Rigorously we would have to ensure the existence of the limit. When $x = 0$ the integral diverges just as the Dirac delta function. This reasing makes the expression plausible, but of course in order to establish that it also satisfies the defining property (3.59) we would need a more detailed analysis.

Having seen now how to compute Laplace transforms with their corresponding restrictions in the domain to guarantee their existence, we can study next some further useful properties.

3.2.1 Properties of the Laplace transform

Before we can apply the Laplace transform to solve differential equations we need familiarize ourselves with some of its basic properties:

i) *Linearity*: The Laplace transform acts linearly

$$\mathcal{L}(\lambda u + \kappa v)(x) = \lambda \mathcal{L}(u)(x) + \kappa \mathcal{L}(v)(x), \qquad \lambda, \kappa \in \mathbb{C}, \tag{3.61}$$

which follows trivially from the definition (3.3).

ii) *Translation*: Unlike as for the Fourier transform the Laplace transform for the shifted function $v(x) = u(x + \Delta)$ with $\Delta \in \mathbb{R}$ is not very useful, as the Δ-shift can not be absorbed into the integration limits such that the resulting term is not a simple one in form of Laplace transforms. More useful is to compute the Laplace transform of $v(x) = e^{x\Delta} u(x)$

$$\mathcal{L}v(x) = \int_0^{\infty} e^{t\Delta} u(t) e^{-tx} dt = \int_0^{\infty} u(t) e^{-t(x-\Delta)} dt = \mathcal{L}u(x-\Delta). \tag{3.62}$$

iii) *Scaling*: The Laplace transform for the scaled function $v(x) = u(\lambda x)$ with $\lambda \in \mathbb{R}^+$ is

$$\mathcal{L}v(x) = \int_0^\infty u(\lambda t)e^{-tx}dt = \frac{1}{\lambda}\int_0^\infty u(t)e^{-xt/\lambda}dt = \frac{1}{\lambda}\mathcal{L}u(x/\lambda). \tag{3.63}$$

Here there is no problem to absorb the scale factor into the integration limits.

iv) *Derivative of u*: The Laplace transform for the derivative $u'(x)$ of the function $u(x)$ is

$$\mathcal{L}u'(x) = \int_0^\infty u'(t)e^{-tx}dt = u(t)e^{-tx}\big|_0^\infty + x\int_0^\infty u(t)e^{-tx}dt = x\mathcal{L}u(x) - u(0), \tag{3.64}$$

where we integrated by parts and assumed that $u(t)$ decays at infinity such that $\lim_{t\to\infty} u(t)e^{-tx} = 0$. Formula (3.64) is easily generalized by integrating n times by parts to

$$\mathcal{L}u^{(n)}(x) = x^n\mathcal{L}u(x) - \sum\nolimits_{k=0}^{n-1} x^{n-k-1}u^{(k)}(0). \tag{3.65}$$

We leave this as an exercise. Clear for $n = 1$ the general formula (3.65) reduces to (3.64).

v) *Derivative of $\mathcal{L}u$*: The derivative of the Laplace transform $\mathcal{L}u(x)$ is

$$\frac{d}{dx}\mathcal{L}u(x) = \frac{d}{dx}\left(\int_0^\infty u(t)e^{-tx}dt\right) = -\int_0^\infty tu(t)e^{-tx}dt = -\mathcal{L}v(x), \tag{3.66}$$

with $v(t) = tu(t)$.

As for the Fourier transform an important property is the Laplace transform of the convolution.

Lemma 4: *The Laplace transform of the convolution of the two functions u and v, i.e. $u \star v(x)$ equals the product of the Laplace transforms these functions*

$$\mathcal{L}(u \star v)(x) = (\mathcal{L}u)(x)(\mathcal{L}v)(x). \tag{3.67}$$

Proof : The proof is similar as the corresponding one for the Fourier transform, but as a difference we start now from the right hand side of the equation (3.67). By definition of the Laplace transform we have

$$\begin{aligned} &(\mathcal{L}u)(x)(\mathcal{L}v)(x) \qquad (3.68)\\ &= \int_0^\infty u(t)e^{-tx}dt\int_0^\infty v(s)e^{-sx}ds \\ &= \int_0^\infty dt\int_0^\infty ds\; u(t)v(s)e^{-x(t+s)}. \end{aligned}$$

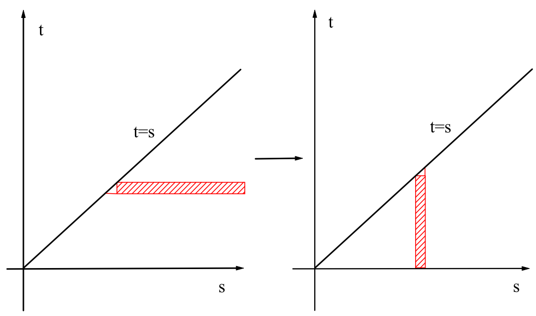


Figure 21: Order of integration change in (3.69).

Shifting now $s \to s - t$ we obtain

$$(\mathcal{L}u)(x)(\mathcal{L}v)(x) = \int_0^\infty dt \int_t^\infty ds\ u(t)v(s-t)e^{-xs}. \tag{3.69}$$

Next we change the order of integration according to $\int_0^\infty dt \int_t^\infty ds \to \int_0^\infty ds \int_0^s dt$. Figure 21 provides an illustration of the fact that in order to cover the entire integration area we can either sum up horizontal or vertical slices. We then obtain

$$\begin{aligned}
(\mathcal{L}u)(x)(\mathcal{L}v)(x) &= \int_0^\infty ds \int_0^s dt\ u(t)v(s-t)e^{-xs} && (3.70)\\
&= \int_0^\infty ds \left(\int_{-\infty}^\infty dt\ u(t)v(s-t) \right) e^{-xs} && (3.71)\\
&= \int_0^\infty ds (u \star v)(s) e^{-xs} && (3.72)\\
&= \mathcal{L}(u \star v)(x). && (3.73)
\end{aligned}$$

In going from (3.70) to (3.71) we have extended integration limits to $\pm\infty$, by using the fact that functions for which we compute Laplace transforms are switched on at $t = 0$, i.e. $u(t) = v(t) = 0$ for $t < 0$. In the step from (3.71) to (3.72) we simply used the definition of the convolution (3.20). $\square$

3.2.2 The inverse of the Laplace transform

For the Fourier transform we have briefly eluded to the notion of the inverse transform. For Laplace transforms this is more complicated and we need to elaborate on their computation in more detail.

Definition: *Suppose that* v(x)=$\mathcal{L}$u(x), *then the* inverse $\mathcal{L}^{-1}$ *of the Laplace transform is the transformation*

$$\mathcal{L}^{-1}v(x) = \mathcal{L}^{-1}\left[\mathcal{L}u(x)\ \right] = u(x). \tag{3.74}$$

Next we have to see how to compute the inverse explicitly.

Lemma 5: *The inverse of the Laplace transform*

$$\mathcal{L}u(x) = v(x) \quad \text{for } x > \alpha \tag{3.75}$$

can be computed by the Bromwich integral

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} v(t)e^{tx}dt \quad \text{for } \gamma > \alpha. \tag{3.76}$$

The contour of the integral is indicated as the vertical line in figure 22 for $r \to \infty$.

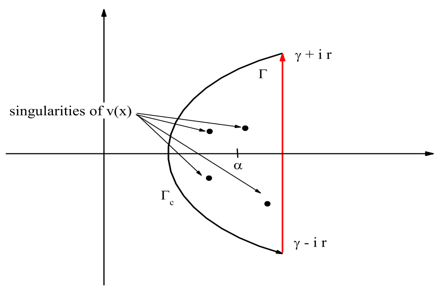


Figure 22: Contour of the Bromwich integral.

Proof : We suppose that $\mathcal{L}u(x) = v(x)$ exists when $x > \alpha$. Extracting now the exponential growth from $u(x)$ by introducing a new function $g(x)$ then we deduce that for $u(x) = e^{\gamma x} g(x)$ the Laplace transform $\mathcal{L}g(x)$ exists for $\gamma > \alpha$. This follows from $|g(x)| = |u(x)e^{-\gamma x}| \leq \mu \left|e^{-\gamma x}\right| |e^{\alpha x}| = \mu e^{(\alpha-\gamma)x}$such that the exponential growth of $g(x)$ is $\alpha - \gamma$ with $\mu \geq 1$. Using the definition of the Dirac delta function (3.59) we can rewrite the function $g(x)$ as

$$g(x) = \int_{-\infty}^{\infty} g(t)\delta(x-t)dt. \tag{3.77}$$

Using now the integral representation (3.60) of the Dirac delta function

$$\delta(x-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(x-t)}d\omega \tag{3.78}$$

we can express $g(x)$ in (3.77) as

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}d\omega \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}d\omega \int_{0}^{\infty} g(t)e^{-i\omega t}dt, \tag{3.79}$$

where in the last equality we simply used the fact that $g(t) = 0$ for $t < 0$. Using now that $u(x) = e^{\gamma x}g(x)$ we obtain

$$u(x) = \frac{e^{\gamma x}}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}d\omega \int_{0}^{\infty} u(t)e^{-\gamma t}e^{-i\omega t}dt. \tag{3.80}$$

Changing variables by $s = \gamma + i\omega$ we compute

$$u(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx}ds \int_{0}^{\infty} u(t)e^{-st}dt = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx}ds\mathcal{L}u(s), \tag{3.81}$$

which is what we wanted to show.□

Let us now see for some concrete examples of how these integrals may be evaluated.
Example 1: We know already from example 4 of the previous section that

$$u(x) = \begin{cases} \sin\lambda x & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \qquad \Leftrightarrow \qquad \mathcal{L}u(x) = \frac{\lambda}{\lambda^2 + x^2} \qquad \text{for } x > 0. \tag{3.82}$$

with $\lambda \in \mathbb{R}$. In other words we know that

$$v(x) = \frac{\lambda}{\lambda^2 + x^2} \qquad \Leftrightarrow \qquad \mathcal{L}^{-1}v(x) = \sin\lambda x. \tag{3.83}$$

Let us compute the inverse of $v(x)$ by using the Bromwich integral.

Solution : The Bromwich integral (3.76) for the function of $v(x)$ is

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\lambda}{\lambda^2 + t^2}e^{tx}dt \qquad \text{for } \varepsilon > 0. \tag{3.84}$$

Provided that the integral over the semi-circle $\Gamma_c : z = \varepsilon + re^{i\theta}$ as indicated in figure 22 with $\gamma \to \varepsilon$, vanishes, we obtain

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i}\oint_\Gamma \frac{\lambda}{(z-i\lambda)(z+i\lambda)}e^{xz_0}dz. \tag{3.85}$$

Since the two simple order poles are on the imaginary axis and $\varepsilon > 0$ the Residue theorem[5] gives

$$\mathcal{L}^{-1}v(x) = \frac{2\pi i}{2\pi i}\operatorname*{Res}_{z_0=\pm i\lambda}\frac{e^{xz_0}}{(z_0-i\lambda)(z_0+i\lambda)}. \tag{3.87}$$

Computing the Residues[6] yields the expected answer

$$\mathcal{L}^{-1}v(x) = \lambda\frac{e^{i\lambda x}}{2i\lambda} + \frac{\lambda e^{-i\lambda x}}{-2i\lambda} = \sin\lambda x. \tag{3.90}$$

We still have to show that the integral over the semi-circle Γ_c vanishes. This means we want to establish

$$\oint_{\Gamma_c}\frac{\lambda}{\lambda^2+z^2}e^{xz}dz = 0 \quad \text{where } \Gamma_c : z = \varepsilon + re^{i\theta}, \frac{\pi}{2} \le \theta \le \frac{3}{2}\pi. \tag{3.91}$$

We estimate

$$\lambda\left|\oint_{\Gamma_c}\frac{e^{xz}}{\lambda^2+z^2}dz\right| = \lambda\left|\int_{\pi/2}^{3\pi/2}\frac{e^{\varepsilon x}e^{xre^{i\theta}}}{\lambda^2+(\varepsilon+re^{i\theta})^2}re^{i\theta}d\theta\right| \tag{3.92}$$

$$\le \lambda e^{\varepsilon x}\int_{\pi/2}^{3\pi/2}\frac{\left|e^{xre^{i\theta}}\right|\left|re^{i\theta}\right|}{\left|\lambda^2+(\varepsilon+re^{i\theta})^2\right|}d\theta \tag{3.93}$$

$$\le \lambda e^{\varepsilon x}\int_{\pi/2}^{3\pi/2}\frac{r}{\lambda^2+r^2}d\theta = \lambda e^{\varepsilon x}\frac{r}{\lambda^2+r^2} \to 0 \quad \text{for } r \to \infty.$$

In (3.93) we have used

$$\begin{aligned}
\left|re^{i\theta}\right| &= r \\
\left|e^{xre^{i\theta}}\right| &= \left|e^{xr\cos\theta}e^{ixr\sin\theta}\right| = e^{xr\cos\theta} \le 1 \quad \text{for } \frac{\pi}{2} \le \theta \le \frac{3}{2}\pi \\
\left|\lambda^2 + \left(\varepsilon + re^{i\theta}\right)^2\right| &> \lambda^2 + r^2
\end{aligned}$$

[5] Recall the Residue theorem from Mathematical Methods I: Let $f(z)$ be an analytic function on a simply connected domain D, except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ and let Γ be some closed positively oriented curve in D, which is piecewise smooth and with $z_1 \notin \Gamma$. Then

$$\oint_\Gamma f(z)dz = 2\pi i\sum\nolimits_{i=1}^{n}\operatorname*{Res}_{z_0=z_i} f(z_0). \tag{3.86}$$

[6] Recall the definition of a Residue from Mathematical Methods I: Suppose there is a Laurent expansion of $f(z)$ about z_0

$$f(z) = \sum\nolimits_{n=-\infty}^{\infty} a_n(z-z_0)^n. \tag{3.88}$$

Then the coefficient

$$a_{-1} = \frac{1}{2\pi i}\oint_\Gamma f(z)dz \tag{3.89}$$

is called the <u>*residue*</u> of $f(z)$ at $z = z_0$.

Example 2: We know already from example 3 of the previous section that

$$u(x) = \begin{cases} e^{\beta x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \qquad \Leftrightarrow \qquad \mathcal{L}u(x) = \frac{1}{x-\beta} \quad \text{for } \ x > \beta. \tag{3.94}$$

with $\beta \in \mathbb{R}$. In other words we know that

$$v(x) = \frac{1}{x-\beta} \qquad \Leftrightarrow \qquad \mathcal{L}^{-1}v(x) = e^{\beta x}. \tag{3.95}$$

Solution : Now the Bromwich integral (3.76) gives

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{t-\beta}e^{tx}dt \qquad \text{for } \mu > \beta \tag{3.96}$$

Provided that the integral over the semi-circle $\Gamma_c : z = \mu + re^{i\theta}$ vanishes (we leave it as an exercise to show this), we obtain

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{\lambda}{z-\beta}e^{xz}dz = \frac{2\pi i}{2\pi i}\operatorname*{Res}_{z_0=\beta}\frac{e^{xz_0}}{z_0-\beta} = e^{\beta x}. \tag{3.97}$$

We have seen that it is far easier to compute the Laplace transform rather than its inverse as the above integrals are relatively involved to compute. It is therefore desirable to have an alternative simpler method at hand. In fact, we knew already the answer in our examples because the functions for which we wanted to compute the Laplace transforms resulted already as a Laplace transform of some other function. We may use this as a general principle and extract these answer from tables of Laplace transforms. Indeed, whenever we have a function of polynomial form $u(x) = g(x)/h(x)$ this can be achieved in a very systematic manner by expanding $u(x)$ in partial fractions (recall Calculus I). Suppose that the degree $g(x)$ in x is lower than the degree of $h(x)$ in x we can write

$$u(x) = \sum\nolimits_n \sum\nolimits_m \frac{c_{n,m}}{(x-\alpha_n)^m}. \tag{3.98}$$

Here the α_n are the roots of $h(x)$ and m are their mulitiplicities. The Laplace transform of each individual term on the right hand side in (3.98) are easily computed and since the Laplace operator is a linear operator (3.61), we know the Laplace transform of $u(x)$. Let us work out some examples to see what this means.
Example 3: Compute the inverse Laplace transform of the function

$$u(x) = \frac{\lambda^2}{x(x^2+\lambda^2)}. \tag{3.99}$$

with $\lambda \in \mathbb{R}$.

Solution : Expanding $u(x)$ gives

$$u(x) = \frac{\lambda^2}{x(x^2+\lambda^2)} = \frac{Ax+B}{x^2+\lambda^2} + \frac{C}{x} = \frac{Ax^2+Bx+Cx^2+C\lambda^2}{x(x^2+\lambda^2)} \tag{3.100}$$

Comparing coefficients of powers in x yields

$$A + C = 0, \qquad B = 0, \qquad C = 1 \qquad \Rightarrow \quad A = -1. \tag{3.101}$$

This means we can write $u(x)$ as

$$u(x) = -\frac{x}{x^2 + \lambda^2} + \frac{1}{x}, \tag{3.102}$$

such that

$$\mathcal{L}^{-1}(u) = \mathcal{L}^{-1}\left(\frac{1}{x}\right) - \mathcal{L}^{-1}\left(\frac{x}{x^2 + \lambda^2}\right). \tag{3.103}$$

From (3.43) we know that

$$\mathcal{L}^{-1}\left(\frac{1}{x}\right) = H(x). \tag{3.104}$$

Furthermore, from our previous example 4 (3.55) we know that

$$\mathcal{L}(\sin \lambda x) = \frac{\lambda}{\lambda^2 + x^2} \qquad \text{for } x > 0. \tag{3.105}$$

Taking now $u(x) = \sin \lambda x$ in the relation for the Laplace transform of the derivative (3.64), we obtain with $u'(x) = \lambda \cos \lambda x$, such that

$$\mathcal{L}(\lambda \cos \lambda x) = \frac{\lambda x}{\lambda^2 + x^2}. \tag{3.106}$$

Therefore we obtain

$$\mathcal{L}^{-1}\left(\frac{x}{x^2 + \lambda^2}\right) = \cos \lambda x. \tag{3.107}$$

Assembling (3.103), (3.104) and (3.107) this gives as a final answer

$$\mathcal{L}^{-1}\left(\frac{\lambda^2}{x(x^2 + \lambda^2)}\right) = H(x) - \cos \lambda x. \tag{3.108}$$

Example 4: Compute the inverse Laplace transform of the function

$$u(x) = \frac{x}{(x+\lambda)(x+\mu)} \qquad \text{for } \lambda \neq \mu; \lambda, \mu \in \mathbb{R}. \tag{3.109}$$

Solution : Again we start by a partial fraction expansion of $u(x)$

$$u(x) = \frac{x}{(x+\lambda)(x+\mu)} = \frac{A}{x+\lambda} + \frac{B}{x+\mu} = \frac{A(x+\mu) + B(x+\lambda)}{(x+\lambda)(x+\mu)}. \tag{3.110}$$

Comparing coefficients of powers in x yields

$$A + B = 1, \qquad A\mu + B\lambda = 0, \qquad \Rightarrow \quad A = \frac{\lambda}{\lambda - \mu}, B = \frac{\mu}{\mu - \lambda} \tag{3.111}$$

This means we can write $u(x)$ as

$$u(x) = \frac{1}{\lambda - \mu}\left(\frac{\lambda}{x+\lambda} - \frac{\mu}{x+\mu}\right), \tag{3.112}$$

such that

$$\mathcal{L}^{-1}(u) = \frac{1}{\lambda - \mu}\left[\mathcal{L}^{-1}\left(\frac{\lambda}{x+\lambda}\right) - \mathcal{L}^{-1}\left(\frac{\mu}{x+\mu}\right)\right]. \tag{3.113}$$

From (3.50) we know that

$$\mathcal{L}^{-1}\left(\frac{1}{x+\beta}\right) = e^{-\beta x} \tag{3.114}$$

Assembling this yields the final answer

$$\mathcal{L}^{-1}\left(\frac{x}{(x+\lambda)(x+\mu)}\right) = \frac{1}{\lambda - \mu}\left[\lambda e^{-\lambda x} - \mu e^{-\mu x}\right]. \tag{3.115}$$

Now we have collected enough techniques to apply the Laplace transform for some concrete boundary value problems.

3.2.3 Applications of the Laplace transform

Just like the Fourier transform we can use the Laplace transform to transform some difficult problems to easier ones. In particular, we can reduce the order of differential equations.

A first order differential equation with given boundary condition Solve the differential equation

$$u'(x) + u(x) = H(x) - H(x-1) \tag{3.116}$$

with boundary condition $u(0) = 1$.

Solution : We start by acting with the Laplace operator on equation (3.116)

$$\mathcal{L}u'(x) + \mathcal{L}u(x) = \mathcal{L}H(x) - \mathcal{L}H(x-1) = \frac{1}{x}(1 - e^{-x}) \tag{3.117}$$

Using the property (3.64) $\mathcal{L}u'(x) = x\mathcal{L}u(x) - u(0)$ and invoking the boundary condition $u(0) = 1$ gives

$$\mathcal{L}u(x)(1+x) - 1 = \frac{1}{x}(1 - e^{-x}). \tag{3.118}$$

Solving this equation for $\mathcal{L}u(x)$ yields

$$\mathcal{L}u(x) = \frac{1}{1+x}\left[\frac{1}{x}(1 - e^{-x}) + 1\right] = \frac{1}{x} - e^{-x}\left(\frac{1}{x} - \frac{1}{1+x}\right). \tag{3.119}$$

Using the Laplace transforms for the functions $v(x) = \delta(x-1)$, $H(x)$, $u(x) = e^{-x}$

$$\mathcal{L}v(x) = e^{-x}, \quad \mathcal{L}H(x) = \frac{1}{x}, \quad \text{and} \quad \mathcal{L}u(x) = \frac{1}{1+x}, \tag{3.120}$$

we can rewrite (3.119) as

$$\mathcal{L}u(x) = \mathcal{L}H(x) - \mathcal{L}v(x)\mathcal{L}H(x) + \mathcal{L}v(x)\mathcal{L}u(x). \tag{3.121}$$

By means of lemma 5 follows

$$\mathcal{L}u(x) = \mathcal{L}H(x) - \mathcal{L}\left[v \star H\right](x) + \mathcal{L}\left[v \star u\right](x), \tag{3.122}$$

such that when acting with $\mathcal{L}^{-1}$ on this equation we obtain

$$\begin{aligned} u(x) &= H(x) - v \star H(x) + v \star u(x) && (3.123) \\ &= H(x) - \int_0^\infty \delta(t-1)H(x-t)dt + \int_0^\infty \delta(t-1)e^{-(x-t)}dt && (3.124) \\ &= \begin{cases} 1 & \text{for} \quad 0 < x < 1 \\ e^{-(x-1)} & \text{for} \quad 1 < x < \infty \end{cases} && (3.125) \end{aligned}$$

The harmonic oscillator A point particle of mass m is fixed on a spring with spring constant κ. Neglecting friction Newton's second law describes the motion of this particle as

$$m\ddot{x}(t) + \kappa x(t) = 0, \tag{3.126}$$

where x is the vertical displacement of the particle as a function of time t. Solve this equation using Laplace transforms with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$.

Solution : Once again we start by acting with the Laplace operator on the equation (**??**)

$$m\mathcal{L}\ddot{x}(t) + \kappa\mathcal{L}x(t) = 0 \tag{3.127}$$

Using the property (3.65) for $n = 2$, i.e. $\mathcal{L}\ddot{x}(t) = t^2\mathcal{L}x(t) - tx(0) - \dot{x}(0)$, and invoking the boundary conditions gives

$$mt^2\mathcal{L}x(t) - mtx_0 + \kappa\mathcal{L}x(t) = 0. \tag{3.128}$$

Solving this equation for $\mathcal{L}x(t)$ yields

$$\mathcal{L}x(t) = \frac{mtx_0}{mt^2 + \kappa} = x_0\frac{t}{t^2 + \omega^2}, \tag{3.129}$$

where we introduced the quantity (frequency) $\omega^2 := \kappa/m$. Therefore we obtain

$$x(t) = x_0\mathcal{L}^{-1}\left(\frac{t}{t^2 + \omega^2}\right) = x_0 \cos\omega x. \tag{3.130}$$

The last equality follows from (3.107). This is of course the answer we expect.

The wave equation with boundary conditions Solve the wave equation

$$\phi_{xx}(x,t) - \phi_{tt}(x,t) = 0 \tag{3.131}$$

subject to the initial conditions $\phi(x,0) = \sin x$ for $0 < x < \pi$, $\phi_t(x,0) = 0$ and $\phi(0,t) = \phi(\pi,t) = 0$ for $t > 0$.

Solution : We start by differentiating twice the Laplace transform in t of $\phi(x,t)$

$$\frac{d^2}{dx^2}\mathcal{L}_t\phi(x,t) = \mathcal{L}_t\phi_{xx}(x,t) = \mathcal{L}_t\phi_{tt}(x,t), \tag{3.132}$$

where in the last equality we used the wave equation. Using the property (3.65) for $n = 2$

$$\mathcal{L}_t\phi_{tt}(x,t) = t^2\mathcal{L}_t\phi(x,t) - t\phi(x,0) - \phi_t(x,0) \tag{3.133}$$

and the initial conditions we obtain a linear second order differential for $\mathcal{L}_t\phi(x,t)$

$$\frac{d^2}{dx^2}\mathcal{L}_t\phi(x,t) = t^2\mathcal{L}_t\phi(x,t) - t\sin x \quad \text{for} \quad 0 < x < \pi. \tag{3.134}$$

This may be solved by

$$\mathcal{L}_t\phi(x,t) = A(t)\sin x + B(t)\cos x. \tag{3.135}$$

From $\phi(0,t) = 0$ follow $B(t) = 0$. Substituting (3.135) into (3.134) gives then

$$-A(t)\sin x = t^2A(t)\sin x - t\sin x \quad \Rightarrow \quad A(t) = \frac{t}{t^2+1}. \tag{3.136}$$

Therefore

$$\mathcal{L}_t\phi(x,t) = \frac{t}{t^2+1}\sin x \tag{3.137}$$

and hence

$$\phi(x,t) = \sin x\mathcal{L}_t^{-1}\frac{t}{t^2+1} \quad = \sin x\cos t. \tag{3.138}$$

In the last equality we used (3.107) for $\lambda = 1$.

Solution of integral equations Solve the Volterra integral equation

$$\phi(x) + \int_0^x K(x-t)\phi(t)dt = f(x) \tag{3.139}$$

for $\phi(x)$ in terms of the Laplace transforms of the function $f(x)$ and $K(x)$.

We may rewrite (3.139) as

$$\phi(x) + \int_0^\infty H(x-t)K(x-t)\phi(t)dt = f(x). \tag{3.140}$$

We recognize that the integral is in fact a convolution, such that (3.140) may be expressed as

$$\phi(x) + v * \phi(x) = f(x), \tag{3.141}$$

with $v(x) = H(x)K(x)$. Acting on this equation with the Laplace operator and making use of lemma 4 gives

$$\mathcal{L}\phi(x) + \mathcal{L}K(x)\mathcal{L}\phi(x) = \mathcal{L}f(x). \tag{3.142}$$

Solving (3.142) for $\mathcal{L}\phi(x)$ yields

$$\mathcal{L}\phi(x) = \frac{\mathcal{L}f(x)}{1+\mathcal{L}K(x)}, \tag{3.143}$$

and hence

$$\phi(x) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}f}{1+\mathcal{L}K}\right)(x), \tag{3.144}$$

Historical remarks and references

There are plenty of books on complex analysis. Here are some you may find useful:

- Complex variables and their application by A.D. Orborne, library location 515.9 OSB
- Fundamentals of Complex Analysis by E.B. Saff, A.D. Snider, library location 515. SAF
- Applied Complex Analysis, N.H. Asmar, G.C. Jones, library location 515.9 ASM
- Theory and Problems of Complex Variables, M.R. Spiegel, library location 515.9 SPI
- Fundamentals of Differential Equations and Boundary Value Problems, R. Kent Nagle, E. B. Saff, library location 515.35 NAG

[1] Carl Friedrich Gauß (1777-1855) was a German mathematician who is sometimes called the "prince of mathematics." He was a prodigious child, at the age of three informing his father of an arithmetical error in a complicated payroll calculation and stating the correct answer. In school, when his teacher gave the problem of summing the integers from 1 to 100 (an arithmetic series $\sum_{k=1}^{n} k = n(n+1)/2$) to his students, Gauß immediately wrote down the correct answer 5050 on his slate.

Figure 23: C.F. Gauß

At the age of 19, Gauß demonstrated a method for constructing a heptadecagon using only a straightedge and compass which had eluded the Greeks. Gauß also showed that only regular polygons of a certain number of sides could be obtained in that manner (a heptagon, for example, could not be constructed.)

Gauß proved the fundamental theorem of algebra (Every polynomial equation having complex coefficients and degree has at least one complex root.). In fact, he gave four different proofs, the first of which appeared in his dissertation. In 1801, he proved the fundamental theorem of arithmetic (The fundamental theorem of arithmetic states that every positive integer (except the number 1) can be represented in exactly one way apart from rearrangement as a product of one or more primes.).

At the age of 24, Gauß published one of the most brilliant achievements in mathematics, Disquisitiones Arithmeticae (1801). In it, Gauß systematized the study of number theory Gauß proved that every number is the sum of at most three triangular numbers and developed the algebra of congruences.

In 1801, Gauß developed the method of least squares fitting, 10 years before Legendre, but did not publish it. The method enabled him to calculate the orbit of the asteroid Ceres, which had been discovered by Piazzi from only three observations. However, after his independent discovery, Legendre accused Gauß of plagiarism. Gauß published his monumental treatise on celestial mechanics Theoria Motus in 1806. He became interested in the compass through surveying and developed the magnetometer and, with Wilhelm Weber measured the intensity of magnetic forces. With Weber, he also built the first successful telegraph.

Gauß is reported to have said "There have been only three epoch-making mathematicians: Archimedes, Newton and Eisenstein" Most historians are puzzled by the inclusion of Eisenstein in the same class as the other two. There is also a story that in 1807 he was interrupted in the middle of a problem and told that his wife was dying. He is purported to have said, "Tell her to wait a moment until I'm through"

Gauß arrived at important results on the parallel postulate, but failed to publish them. Credit for the discovery of non-Euclidean geometry therefore went to Janos Bolyai and Lobachevsky. However, he did publish his seminal work on differential geometry in Disquisitiones circa superticies curvas. The Gaußian curvature (or "second" curvature) is named for him. He also discovered the Cauchy integral theorem (This is equation (3.86) in the absence of any isolated singularity, i.e. when the right hand side becomes zero.)for analytic functions, but did not publish it.

Gauß reworked and improved papers incessantly, therefore publishing only a fraction of his work, in keeping with his motto "pauca sed matura" (few but ripe). Many of his results were subsequently repeated by others, since his terse diary remained unpublished for years after his death. This diary was only 19 pages long, but later confirmed his priority on many results he had not published. Gauß wanted a heptadecagon placed on his gravestone, but the carver refused, saying it would be indistinguishable from a circle. The heptadecagon appears, however, as the shape of a pedestal with a statue erected in his honor in his home town of Braunschweig.
(Taken from http://scienceworld.wolfram.com/biography/Gauss.html.)

[2] Leonard Euler (1707-1783) was a Swiss mathematician mathematician who was tutored by Johann Bernoulli. He worked at the Petersburg Academy and Berlin Academy of Science. He had a phenomenal memory, and once did a calculation in his head to settle an argument between students whose computations differed in the fiftieth decimal place. Euler lost sight in his right eye in 1735, and in his left eye in 1766. Nevertheless, aided by his phenomenal memory (and having practiced writing on a large slate when his sight was failing him), he continued to publish his results by dictating them. Euler was the most prolific mathematical writer of all times finding time (even with his 13 children) to publish over 800 papers in his lifetime. He won the Paris Academy Prize 12 times. When asked for an explanation why his memoirs flowed so easily in such huge quantities, Euler is reported to have replied that his pencil seemed to surpass him in intelligence. Franç ois Arago said of him "He calculated just as men breathe, as eagles sustain themselves in the air".

Figure 24: L. Euler

Euler systematized mathematics by introducing the symbols e , i , and $f(x)$ for f a function of x. He also made major contributions in optics, mechanics, electricity, and magnetism. He made significant contributions to the study of differential equations. His Introducio in analysin infinitorum (1748) provided the foundations of analysis. He showed that any complex number to a complex power can be written as a complex number, and investigated the beta and gamma functions. He computed the Riemann zeta function to for even numbers.

He also did important work in number theory, proving that that the divergence of the harmonic series implied an infinite number of Primes, factoring the fifth Fermat number (thus disproving

Fermat's conjecture), proving Fermat's lesser theorem, and showing that e was irrational. In 1772, he introduced a synodic coordinates (rotating) coordinate system to the study of the three-body problem (especially the Moon). Had Euler pursued the matter, he would have discovered the constant of motion later found in a different form by Jacobi and known as the Jacobi integral.

Euler also found the solution to the two fixed center of force problem for a third body. Finally, he proved the binomial theorem was valid for any rational exponent. In a testament to Euler's proficiency in all branches of mathematics, the great French mathematician and celestial mechanic Laplace told his students, "Liesez Euler, Liesez Euler, c'est notre maître à tous" ("Read Euler, read Euler, he is our master in everything"
(Taken from http://scienceworld.wolfram.com/biography/Euler.html)

[3] **Georg Friedrich Bernhard Riemann** (1826-1866) was a German mathematician whose profound and novel approaches to the study of geometry laid the mathematical foundation for Albert Einstein's theory of relativity. He also made important contributions to the theory of functions, complex analysis, and number theory.

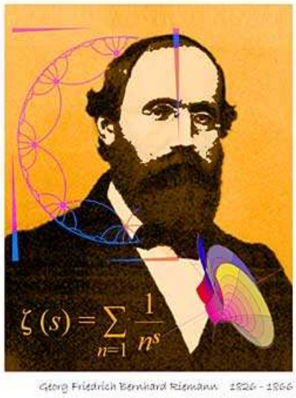


Figure 25: G.F.B. Riemann

Riemann was born into a poor Lutheran pastor's family, and all his life he was a shy and introverted person. He was fortunate to have a schoolteacher who recognized his rare mathematical ability and lent him advanced books to read, including Adrien-Marie Legendre's Number Theory (1830). Riemann read the book in a week and then claimed to know it by heart. He went on to study mathematics at the University of Göttingen in 1846–47 and 1849–51 and at the University of Berlin (now the Humboldt University of Berlin) in 1847–49. He then gradually worked his way up the academic profession, through a succession of poorly paid jobs, until he became a full professor in 1859 and gained, for the first time in his life, a measure of financial security. However, in 1862, shortly after his marriage to Elise Koch, Riemann fell seriously ill with tuberculosis. Repeated trips to Italy failed to stem the progress of the disease, and he died in Italy in 1866.

Riemann's visits to Italy were important for the growth of modern mathematics there; Enrico Betti in particular took up the study of Riemannian ideas. Ill health prevented Riemann from publishing all his work, and some of his best was published only posthumously—e.g., the first edition of Riemann's Gesammelte mathematische Werke (1876; "Collected Mathematical Works"), edited by Richard Dedekind and Heinrich Weber.

Riemann's influence was initially less than it might have been. Göttingen was a small university, Riemann was a poor lecturer, and, to make matters worse, several of his best students died young. His few papers are also difficult to read, but his work won the respect of some of the best mathematicians in Germany, including his friend Dedekind and his rival in Berlin, Karl Weierstrass. Other mathematicians were gradually drawn to his papers by their intellectual depth, and in this way he set an agenda for conceptual thinking over ingenious calculation. This emphasis was taken up by Felix Klein and David Hilbert, who later established Göttingen as a world centre for mathematics research, with Carl Gauss and Riemann as its iconic figures.

In his doctoral thesis (1851), Riemann introduced a way of generalizing the study of polynomial equations in two real variables to the case of two complex variables. In the real case a polynomial equation defines a curve in the plane. Because a complex variable z can be thought of as a pair of real variables x + iy (where i = ?(?1)), an equation involving two complex variables defines a real surface—now known as a Riemann surface—spread out over the plane. In 1851 and in his more widely available paper of 1857, Riemann showed how such surfaces can be classified by a number, later called the genus, that is determined by the maximal number of closed curves that can be drawn on the surface without splitting it into separate pieces. This is one of the first significant uses of topology in mathematics. In 1854 Riemann presented his ideas on geometry for the official postdoctoral qualification at Göttingen; the elderly Gauss was an examiner and was greatly impressed. Riemann argued that the fundamental ingredients for geometry are a space of points (called today a manifold) and a way of measuring distances along curves in the space. He argued that the space need not be ordinary Euclidean space and that it could have any dimension (he even contemplated spaces of infinite dimension). Nor is it necessary that the surface be drawn in its entirety in three-dimensional space. A few years later this inspired the Italian mathematician Eugenio Beltrami to produce just such a description of non-Euclidean geometry, the first physically plausible alternative to Euclidean geometry. Riemann's ideas went further and turned out to provide the mathematical foundation for the four-dimensional geometry of space-time in Einstein's theory of general relativity. It seems that Riemann was led to these ideas partly by his dislike of the concept of action at a distance in contemporary physics and by his wish to endow space with the ability to transmit forces such as electromagnetism and gravitation.

In 1859 Riemann also introduced complex function theory into number theory. He took the zeta function, which had been studied by many previous mathematicians because of its connection to the prime numbers, and showed how to think of it as a complex function. The Riemann zeta function then takes the value zero at the negative integers (the so-called trivial zeros) and also at points on a certain line (called the critical line). Standard methods in complex function theory, due to Augustin-Louis Cauchy in France and Riemann himself, would give much information about the distribution of prime numbers if it could be shown that all the nontrivial zeros lie on this line—a conjecture known as the Riemann hypothesis. All nontrivial zeros discovered thus far have been on the critical line. In fact, infinitely many zeros have been discovered to lie on this line. Such partial results have been enough to show that the number of prime numbers less than any number x is well approximated by x/ln x. The Riemann hypothesis was one of the 23 problems that Hilbert challenged mathematicians to solve in his famous 1900 address, "The Problems of Mathematics." Over the years a growing body of mathematical ideas have built upon the assumption that the Riemann hypothesis is true; its proof, or disproof, would have far-reaching consequences and confer instant renown. Riemann took a novel view of what it means for mathematical objects to exist. He sought general existence proofs, rather than "constructive proofs" that actually produce the objects. He believed that this approach led to conceptual clarity and prevented the mathematician from getting lost in the details, but even some experts disagreed with such nonconstructive proofs. Riemann also studied how functions compare with their trigonometric or Fourier series representation, which led him to refine ideas about discontinuous functions. He showed how complex function theory illuminates the study of minimal surfaces (surfaces of least area that span a given boundary). He was one of the first to study differential equations involving complex variables, and his work led to a profound connection

with group theory. He introduced new general methods in the study of partial differential equations and applied them to produce the first major study of shock waves.
(Taken from http://www.britannica.com/)

[4] **Augustin Cauchy** (1789-1857) was a French mathematician who wrote 789 papers, a quantity exceeded only by Euler and Cayley, which brought precision and rigor to mathematics. He invented the name for the determinant and systematized its study and gave nearly modern definitions of limit, continuity, and convergence. Cauchy founded complex analysis by discovering the Cauchy-Riemann equations (although these had been previously discovered by d'Alembert).

Figure 26: A. Cauchy

Cauchy also presented a mathematical treatment of optics, hypothesized that ether had the mechanical properties of an elasticity medium, and published classical papers on wave propagation in liquids and elastic media. After generalizing Navier's equations for isotropic media, he formulated one for anisotropic media. Cauchy published his first elasticity theory in 1830 and his second in 1836. Both were rather ad hoc and were riddled with problems, and Cauchy proposed a third theory in 1839. Cauchy also studied the reflection from metals and dispersion relationships.

Cauchy extended the polyhedral formula in a paper which was criticized by Malus. His theory of substitutions led to the theory of finite groups. He proved that the order of any subgroup is a divisor of the order of the group. He also proved Fermat's three triangle theorem. He refereed a long paper by Le Verrier on the asteroid Pallas and invented techniques which allowed him to redo Le Verrier's calculations at record speed. He was a man of strong convictions, and a devout Catholic. He refused to take an oath of loyalty, but also refused to leave the French Academy of Science.
(Taken from http://scienceworld.wolfram.com/biography/Cauchy.html)

[5] **Pierre Laplace** (1749-1827) was a French physicist and mathematician who put the final capstone on mathematical astronomy by summarizing and extending the work of his predecessors in his five volume Mécanique Céleste (Celestial Mechanics) (1799-1825). This work was important because it translated the geometrical study of mechanics used by Newton to one based on calculus, known as physical mechanics. In Mécanique Céleste, Laplace proved the dynamical stability of the solar system (with tidal friction ignored) on short time scales. On long time scales, however, this assertion was proven false in the early 1990s. Laplace solved the libration of the Moon. In this work, he frequently omitted derivations, leaving only results with the remark "il est aisé à voia" (it is easy to see). It is said that he himself could not always fill in the derivations later without days of work. For a revealing quote, see the remark made by Laplace's translator Bowditch. After reading Mécanique céleste, Napoleon is said to have questioned Laplace on his neglect to mention God. In stark contrast to Newton's view on the subject, Laplace replied that he had no need for that hypothesis.

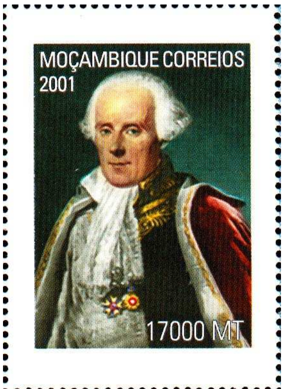


Figure 27: P. Laplace

Laplace also systematized and elaborated probability theory in "Essai Philosophique sur les Probabilités" (Philosophical Essay on Probability, 1814). He was the first to publish the value of the Gaussian integral, . He studied the Laplace transform, although Heaviside developed the techniques fully. He proposed that the solar system had formed from a rotating solar nebula with rings breaking off and forming the planets. He discussed this theory in Exposition de système du monde (1796). He pointed out that sound travels adiabatically, accounting for Newton's too small value. Laplace formulated the mathematical theory of interparticulate forces which could be applied to mechanical, thermal, and optical phenomena. This theory was replaced in the 1820s, but its emphasis on a unified physical view was important.

With Lavoisier, whose caloric theory he subscribed to, he determined specific heats for many substances using a calorimeter of his own design. Laplace borrowed the potential concept from Lagrange, but brought it to new heights. He invented gravitational potential and showed it obeyed Laplace's equation in empty space.

After being appointed Minister of the Interior by Napoleon, Laplace was dismissed with the comment that "he carried the spirit of the infinitely small into the management of affairs".

Laplace believed the universe to be completely deterministic.

(Taken from http://scienceworld.wolfram.com/biography/Laplace.html)