Mathematical Methods II

Coursework 1 (Solutions)

DEADLINE: Tuesday 11/03/2014 at 13:00

1) **i)** We first map the plait at the origin to the second plait in the vertical braid by $\boxed{4}$

$$T_1(z) = \lambda z + ib,$$

where $\lambda > 1$ is a scaling factor intruduced because the plaits are getting larger when we move up the braid. Continuing this process to the 35th braid we obtain

$$\begin{aligned} T_1^{34}(z) &= \lambda^{34} z + ib \sum\nolimits_{k=0}^{n-1} \lambda^k \\ &= \lambda^{34} z + ib \left(\frac{\lambda^{34} - 1}{\lambda - 1} \right). \end{aligned}$$

Next we need to rotate the vertical braid by an angle $-\pi/2$. Notice also that the plaits are getting smaller when moving away from the vertical braid. We incorporate this by a scaling factor $\kappa < 1$. Finally we notice that there is a horizontal displacement for each braid away from the origin, say by a distance ρ. Assembling this we can map the plait at the origin to the last plait in the last braid by

$$\kappa e^{-i\pi/2} T_1^{34}(z) + \rho = \kappa e^{-i\pi/2} \left[\lambda^{34} z + ib \left(\frac{\lambda^{34} - 1}{\lambda - 1} \right) \right] + \rho.$$

ii) By a similar reasoning we obtain the 20th plait in the 5th braid by $\boxed{1}$

$$\kappa e^{-i\pi 4/24} \left[\lambda^{19} z + ib \left(\frac{\lambda^{19} - 1}{\lambda - 1} \right) \right] + \tilde{\rho}.$$

2) The cross ration is a linear fractional transformation. Recall from the lecture that one can decompose the linear fractional transformation $\boxed{5}$

$$T(z) = \frac{az + b}{cz + d} \qquad \text{for } ad - bc \neq 0; a, b, c, d \in \mathbb{C}$$

into

$$\begin{aligned} T^{c=0}(z) &= f_T^{b/d} \circ f_R^{a/d}(z), \\ T^{c\neq 0}(z) &= f_T^{a/c} \circ f_R^{(bc-ad)/c} \circ f_I \circ f_T^{d} \circ f_R^{c}(z). \end{aligned}$$

where f_T, f_R, f_I are the translation, rotation and inversion maps.

- The difference $(z_i - z_j)$ is an invariant of f_T^Δ :

$$f_T^\Delta(z_i - z_j) = z_i + \Delta - z_j - \Delta = z_i - z_j$$

  Since the cross ratio is a product and ratio of such differences it is also an invariant of the translation map

$$f_T^\Delta(T_c) = T_c.$$

- The ratio $(z_i - z_j)/(z_k - z_l)$ is an invariant of f_R^λ :

$$f_R^\lambda\left(\frac{z_i - z_j}{z_k - z_l}\right) = \frac{\lambda z_i - \lambda z_j}{\lambda z_k - \lambda z_l} = \frac{z_i - z_j}{z_k - z_l}$$

  Since the cross ratio is a product of two such ratios it is also an invariant of the rotation map

$$f_R^\lambda(T_c) = T_c.$$

- T_c is an invariant of f_I :

$$f_I\left(T_c\right) = \frac{(z_4^{-1} - z_1^{-1})(z_2^{-1} - z_3^{-1})}{(z_4^{-1} - z_3^{-1})(z_2^{-1} - z_1^{-1})} = \frac{z_4 z_1 (z_4^{-1} - z_1^{-1})(z_2^{-1} - z_3^{-1}) z_2 z_3}{z_4 z_3 (z_4^{-1} - z_3^{-1})(z_2^{-1} - z_1^{-1}) z_2 z_1} = T_c$$

  Since $T(z)$ is composed of f_T, f_R, f_I and T_c is an invariant of all individual transformations, it must also be an invariant of $T(z)$.

3) First we map the exterior of the unit circle in the first quadrant onto the exterior of the unit circle in the first and second quadrant by means of $\boxed{15}$

$$\tilde{w} = \tilde{f}(z) = z^2.$$

Next we map the exterior of the unit circle in the first and second quadrant onto the upper half plane

$$\hat{w} = \hat{f}(\tilde{w}) = \tilde{w} + \frac{1}{\tilde{w}}.$$

Therefore the map of the exterior of the unit circle in the first quadrant onto the upper half plane is

$$f(z) = \hat{f} \circ \tilde{f}(z) = z^2 + \frac{1}{z^2}.$$

Let us verify this in detail and see where the boundary of the exterior of the unit circle in the first quadrant is mapped to:

$$\begin{aligned}
f(iy) &= -\left(y^2 + \frac{1}{y^2}\right) \in (-\infty, -2] \qquad &&\text{for } 1 \leq y < \infty, \\
f(e^{i\theta}) &= 2\cos 2\theta \in [-2, 2] &&\text{for } 0 \leq \theta \leq \frac{\pi}{2}, \\
f(x) &= \left(x^2 + \frac{1}{x^2}\right) \in [2, \infty) &&\text{for } 1 \leq x < \infty,
\end{aligned}$$

This means the boundary is mapped onto the real axis. We could still have the situation that the exterior of the unit circle is mapped to the lower half plane. It suffices to check for one point. For instance $z = 2e^{i\pi/4}$

$$f(2e^{i\pi/4}) = 4e^{i\pi/2} + \frac{1}{4e^{i\pi/2}} = 4i + \frac{1}{4i} = \frac{-16+1}{4i} = \frac{15}{4}i,$$

which is in the upper half plane. The Riemann mapping theorem guarantees the existence of this map and in addition that it is one-to-one.

4) The exterior angles at w_1, w_2, w_3 are $3\pi/4, \pi/2$ and $3\pi/4$. According to the Schwarz-Christoffel theorem the map is therefore given as $\boxed{10}$

$$f'(z) = c(z+1)^{-3/4}(z-1)^{-1/2}.$$

Therefore

$$f(z) = c\int_1^z d\hat{z}(\hat{z}+1)^{-3/4}(\hat{z}-1)^{-1/2} + \tilde{c}.$$

We have

$$f(1) = w_2 = 0 \qquad \Rightarrow \qquad \tilde{c} = 0.$$

We also have

$$\begin{aligned} f(-1) &= c\int_1^{-1} d\hat{z}(\hat{z}+1)^{-3/4}(\hat{z}-1)^{-1/2} \\ &= c(-1)\int_{-1}^{1} d\hat{z}(\hat{z}+1)^{-3/4}(1-\hat{z})^{-1/2}(-1)^{-1/2} \\ &= ic\alpha = w_1 = ia \end{aligned}$$

and therefore

$$c = a/\alpha.$$

This means the transformation which maps the upper half plane onto the specified isosceles right triangle is

$$f(z) = \frac{a}{\alpha}\int_1^z d\hat{z}(\hat{z}+1)^{-3/4}(\hat{z}-1)^{-1/2}.$$

The Schwarz-Christoffel theorem guarantees the existence of this map.

In addition one may verify that

$$w_3 = \frac{a}{\alpha}\int_1^\infty d\hat{z}(\hat{z}+1)^{-3/4}(\hat{z}-1)^{-1/2} = a.$$

However, this was not asked.

5) First we express the $\mathrm{arccot}(z)$ in terms of logarithmic functions. We have $\boxed{15}$

$$z = \cot w = \frac{\cos w}{\sin w} = i\frac{e^{iw}+e^{-iw}}{e^{iw}-e^{-iw}} = i\frac{\lambda+\lambda^{-1}}{\lambda-\lambda^{-1}},$$

where $\lambda = e^{iw}$. Solving this equation for λ we obtain

$$z(\lambda^2 - 1) = i(\lambda^2 + 1) \quad \Rightarrow \ \lambda = \sqrt{\frac{z+i}{z-i}} = e^{iw},$$

and therefore

$$w = -i \ln \sqrt{\frac{z+i}{z-i}}.$$

Hence

$$\operatorname{arccot}(z) = \frac{1}{2i} \ln \left(\frac{z+i}{z-i} \right).$$

Next we write

$$z \pm i = |z \pm i| \, e^{i\theta_{1,2}},$$

such that

$$\operatorname{arccot}(z) = \frac{1}{2i} \left[\ln \left| \frac{z+i}{z-i} \right| + i(\theta_1 - \theta_2) \right].$$

This function has branch points at $\pm i$. This makes it convenient to introduce vertical branch cuts using the conventions

$$i) \ -\frac{3\pi}{2} < \theta_{1,2} < \frac{\pi}{2} \quad \text{or} \quad ii) \ -\frac{\pi}{2} < \theta_{1,2} < \frac{3\pi}{2}.$$

Let us discuss convention $i)$ in more detail. There are three region we need to consider separately:

- $(-i\infty, -i)$: In this region both of the functions are continuous and therefore we do not need a branch cut.
- $(-i, i)$: On the right of the imaginary axis we have

$$\theta_1 = -\frac{\pi}{2}, \quad \theta_2 = \frac{\pi}{2} \qquad \Rightarrow \theta_1 - \theta_2 = -\pi.$$

  On the left of the imaginary axis we have

$$\theta_1 = \frac{\pi}{2}, \quad \theta_2 = -\frac{3\pi}{2} \qquad \Rightarrow \theta_1 - \theta_2 = 2\pi.$$

  This means the function $\operatorname{arccot}(z)$ is discontinuous across the line segment $(-i, i)$ and we require a branch cut to make it analytic.
- $(i, i\infty)$: On the right of the imaginary axis we have

$$\theta_1 = \frac{\pi}{2}, \quad \theta_2 = \frac{\pi}{2} \qquad \Rightarrow \theta_1 - \theta_2 = 0.$$

  On the left of the imaginary axis we have

$$\theta_1 = -\frac{3\pi}{2}, \quad \theta_2 = -\frac{3\pi}{2} \qquad \Rightarrow \theta_1 - \theta_2 = 0.$$

  This means the function $\operatorname{arccot}(z)$ is continuous across the halfline (i, ∞) and we do not require any branch cut here.

Thus overall we need a branch cut on the line segment $(-i, i)$ to make the the function $\operatorname{arccot}(z)$ analytic.

The same conclusion is reached by a similar argument for the convention $ii)$.