

Mathematical Methods II

Coursework 1

Hand in the complete solutions to all five questions to the SEMS general office (C109).

DEADLINE: Tuesday 11/03/2014 at 13:00

- 1) [5 marks] The cornrow hairstyle in the figure



Figure 1: Cornrow hairstyle

is made from 25 braids, with each braid consisting of 35 plaits (y-shapes) of height b and width a . Approximate this style in the complex plane and compute the linear fractional transformation

$$w = T(z) = \frac{az + b}{cz + d} \quad \text{for } ad - bc \neq 0; a, b, c, d \in \mathbb{C}$$

- i) mapping the plait at the origin to the end of the last plait at the end the left hand side of the head,
- ii) to the 20th plait in the 5th braid.

- 2) [5 marks] The cross ratio for the points (z_1, z_2, z_3, z_4) is defined as

$$T_c = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)}.$$

Show that the cross ratio is an invariant of the linear fractional transformation, which means T_c for (z_1, z_2, z_3, z_4) is the same as T_c for $(T(z_1), T(z_2), T(z_3), T(z_4))$.

- 3) [15 marks] Construct an analytic function which maps the exterior of the unit disk in the first quadrant in the z -plane

$$\mathcal{W} = \{r, \theta : r \geq 1, 0 \leq \theta < \frac{\pi}{2}\}$$

conformally onto the upper half plane $\text{Im } w \geq 0$. Draw a figure and indicate the corresponding regions including some characteristic points representing part of the boundary. When using composite maps draw the regions and include points for each function used. Which theorem guarantees that such a map exists?

- 4) [10 marks] Determine the Schwarz-Christoffel transformation, which maps the upper half plane onto an isosceles right triangle. Map the points $x_1 = -1$, $x_2 = 1$ and $x_3 \rightarrow \infty$ to $w_1 = ia$, $w_2 = 0$ and $w_3 = a$. Express your result in terms of the quantity

$$\alpha = \int_{-1}^1 d\hat{z} \frac{1}{(1 + \hat{z})^{3/4}(1 - \hat{z})^{1/2}} \approx 4.40976.$$

Draw the corresponding z and w -plane. Which theorem guarantees that such a map exists?

- 5) [15 marks] Find a domain on which the function

$$f(z) = \text{arccot}(z)$$

is single valued and analytic.