Mathematical Methods II

Coursework 1

Hand in the complete solutions to all three questions in the general office by Thursday 01/11/2007 at 17:00

1) (20 marks)

i) Determine the linear fractional transformation in the form

$$T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad-bc \neq 0; a,b,c,d \in \mathbb{C},$$

which maps the points $z_1 = -1, z_2 = -4, z_3 = -3$ in the z-plane onto the points $w_1 \to \infty, w_2 = 0, w_3 = -1/2$ in the w-plane (image plane). Is this map unique?

ii) Compute all fixed points of the map $T(z)$ in i).

iii) Consider a circle in the complex plane with centre on the imaginary axis passing through all fixed points computed in ii). Show that $T(z)$ from i) maps any of these type of circles into itself, i.e. that the image is identical to the circle in the z-plane.

iv) New linear fractional transformation can be constructed from successive actions of $T(z)$ as

$$T_2(z) := T \circ T(z), \quad T_3(z) := T \circ T_2(z), \ldots \quad T_n(z) = T \circ T_{n-1}(z).$$

Since $T(z)$ maps a circle passing through the fixed points onto itself, also $T_n(z)$ maps points on this circle onto points of the original circle, when one identifies the z-plane with the image plane. For $n \to \infty$ all points will be mapped into one of the fixed points. Convince yourself of this fact by computing $T_n(z)$ for $n = 1,2,3,4,5,6$. Subsequently trace the points $z_1 = -3/2 + i3/2$ and $z_2 = -1/2 + i3/2$ along their circles through the fixed points under the action of $T_2(z)$, $T_4(z)$ and $T_6(z)$. Illustrate your result with a picture and indicate the direction in which the points move under the action of $T_n(z)$ by an arrow. Draw several more of such circles, which you may guess without explicit calculation. The resulting picture should resembles a dipole.

2) (10 marks) Use Euler's formula to show that

$$\arcsin(z) = -i \ln\left(iz + \sqrt{1-z^2}\right).$$

Subsequently use the principal branch of the logarithmic function to determine the domain of analyticity for $\arcsin(z)$.

3) (20 marks)

i) A *group* ($\mathbf{g}$,$\circ$) is a set of elements equipped with a binary operation $\circ$, satisfying the follwing axioms:

a) *Closure*: For any two elements $a, b \in \mathbf{g}$ also $a \circ b \in \mathbf{g}$.

b) *Existence of the identity*: For all elements $a \in \mathbf{g}$ there exists an elements $e \in \mathbf{g}$, such that $e \circ a = a \circ e = a$.

c) *Existence of the inverse*: For each elements $a \in \mathbf{g}$ there exists an element $a^{-1} \in \mathbf{g}$, such that $a^{-1} \circ a = a \circ a^{-1} = e$.

d) *Associativity*: For any three elements $a, b, c \in \mathbf{g}$ the relation $(a \circ b) \circ c = a \circ (b \circ c)$ is satisfied.

Verify the statement that the set of all linear fractional transformations is a group.

ii) The group obtained in this way can be represented by 2×2-matrices

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Taking now the binary operation to be a matrix multiplication verify that the product of two matrices

$$T_1 T_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

can be identified with the composition $T_1 \circ T_2(z)$. Verify that the set of 2×2-matrices form a group as defined in i).

Solutions CW1 (2007)

1 $z_1 = -1,\ z_2 = -4,\ z_3 = -3 \ \xrightarrow[T(z)]{}\ w_1 \to \infty,\ w_2 = 0,\ w_3 = -\frac{1}{2}$

The map can be determined from:

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\Rightarrow \quad \frac{0 + \frac{1}{2}}{w + \frac{1}{2}} = \frac{(z+1)(-4+3)}{(z+3)(-4+1)} \quad \Rightarrow \quad \frac{2w+1}{1} = \frac{3z+9}{z+1}$$

$$\Rightarrow \quad 2w = \frac{3z+9}{z+1} - 1 = \frac{3z+9-z-1}{z+1}$$

$$\Rightarrow \quad w = T(z) = \frac{z+4}{z+1}$$

The map is unique by theorem 5. (4)

1 The fixed points have to satisfy

$$T(z) = z = \frac{z+4}{z+1} \quad \Rightarrow \quad z^2 + z = z + 4$$

$\Rightarrow$ $z = \pm 2$ are the two fixed points. (1

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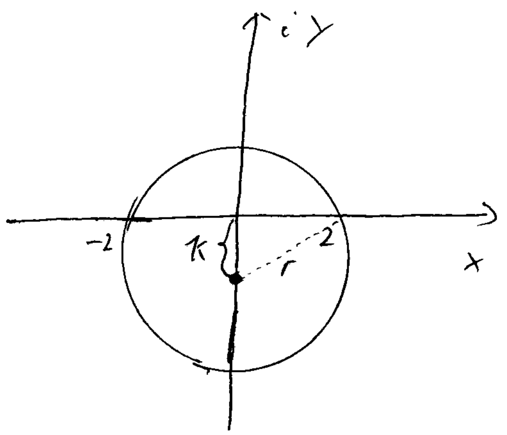

$$\Rightarrow \quad r^2 = k^2 + 2^2$$

$\Rightarrow$ The equation describing the circle is:

$$x^2 + (y - k)^2 = 4 + k^2$$

$$\Leftrightarrow \quad x^2 + y^2 + k^2 - 2ky = 4 + k^2$$

$$\Rightarrow \qquad k = \frac{x^2 + y^2 - 4}{2y}$$

For the points to be mapped into the same circle we need

$$|T(z) - ik|^2 = r^2 = k^2 + 4$$

$$\left|\frac{z+4}{z+1} - ik\right|^2 = \left(\frac{z+4}{z+1} - ik\right)\left(\frac{z^*+4}{z^*+1} + ik\right)$$

$$\frac{z+4}{z+1}\,\frac{z^*+4}{z^*+1}+ik\left(\frac{z+4}{z+1}-\frac{z^*+4}{z^*+1}\right)+k^2$$

$$=\frac{zz^*+4(z+z^*)+16+ik(\cancel{zz^*}+4z^*+z+\cancel{4}-\cancel{zz^*}-4z-z^*-\cancel{4})}{zz^*+(z+z^*)+1}+k^2$$

$$=\frac{zz^*+4(z+z^*)+16+ik\,3(z^*-z)}{zz^*+(z+z^*)+1}+k^2$$

$$=\frac{zz^*+4(z+z^*)+16+6ky}{zz^*+(z+z^*)+1}+k^2 \qquad \because\ (z^*-z)=-2iy$$

$$=\frac{zz^*+4(z+z^*)+16-12+3zz^*}{zz^*+(z+z^*)+1}+k^2 \qquad \because\ \underbrace{x^2+y^2}_{zz^*}-4=2ky$$

$$=4+k^2=|T(z)-ik|^2$$

(7)

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$$T_2=T\circ T(z)=\frac{\frac{z+4}{z+1}+4}{\frac{z+4}{z+1}+1}=\frac{z+4+4z+4}{z+4+z+1}=\frac{5z+8}{2z+5}$$

$$T_3=T\circ T_2(z)=\frac{\frac{5z+8}{2z+5}+4}{\frac{5z+8}{2z+5}+1}=\frac{5z+8+8z+20}{5z+8+2z+5}=\frac{13z+28}{7z+13}$$

$$T_4=T\circ T_3(z)=\frac{\frac{13z+28}{7z+13}+4}{\frac{13z+28}{7z+13}+1}=\frac{13z+28+28z+52}{13z+28+7z+13}=\frac{41z+80}{20z+41}$$

$$T_5=T\circ T_4(z)=\frac{\frac{41z+80}{20z+41}+4}{\frac{41z+80}{20z+41}+1}=\frac{41z+80+80z+164}{41z+80+20z+41}=\frac{121z+244}{61z+121}$$

$$T_6=T\circ T_5(z)=\frac{\frac{121z+244}{61z+121}+4}{\frac{121z+244}{61z+121}+1}=\frac{121z+244+244z+484}{121z+244+61z+121}=\frac{365z+728}{182z+365}$$

The circle passing through ± 2 and $z_1 = -\frac{3}{2} + \frac{3}{2} i$

has its centre at $\left(0, \frac{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 4}{2 \cdot \frac{3}{2}}\right) = \left(0, \frac{1}{6}\right) \Rightarrow r = 4$

The circle passing through ± 2 and $z_2 = -\frac{1}{2} + \frac{3}{2} i$

has its centre at $\left(0, \frac{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 4}{2 \cdot \frac{3}{2}}\right) = \left(0, -\frac{1}{2}\right) \Rightarrow r = 4 \frac{1}{4}$

$$T_2(z_1) = \frac{47}{26} + \frac{27i}{26} = 1,81 + 1,04 i$$

$$T_4(z_1) = \frac{4097 + 243i}{2042} = 2.01 + 0.12 i$$

$$T_6(z_1) = \frac{332147 + 2187i}{165986} = 2.00 + 0.01 i$$

$$T_2(z_2) = \frac{89 + 27i}{50} = 1,78 + 0.54 i$$

$$T_4(z_2) = \frac{7379 + 243i}{3722} = 1,98 + 0,07 i$$

$$T_6(z_2) = \frac{597869 + 2187i}{299210} = 1.998 + 0,01 i$$

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plot picture

1

$\boxed{\Sigma = 20}$

2)

$$\sin z =: w = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \qquad \text{define } y = e^{iz}$$

$$w = \frac{1}{2i}\left(y - y^{-1}\right)$$

$$\Leftrightarrow \quad 2iwy = y^2 - 1 \qquad \Rightarrow \quad y^2 - 2iwy - 1 = 0$$

$$\Rightarrow \qquad y_\pm = iw \pm \sqrt{(iw)^2 + 1}$$

$$= iw \pm \sqrt{1 - w^2}$$

$$\Rightarrow \quad iz = \log\left(iw \pm \sqrt{1-w^2}\right)$$

$$\Rightarrow \quad \underline{\arcsin z = -i \log\left(iz + \sqrt{1-z^2}\right)}$$

3

We can write $\arcsin z = -i \log\left[iz + \exp\left(\frac{1}{2}\log(1-z^2)\right)\right]$

The principle branch has a cut at $(-\infty, 0) \equiv \mathbb{R}^-$. Thus we need to ensure that i) $1 - z^2 \notin \mathbb{R}^-$ and ii) $iz + \exp\left(\frac{1}{2}\log(1-z^2)\right) \notin \mathbb{R}^-$

i) Suppose that $1 - z^2 \in \mathbb{R}$ $\Rightarrow$ $(1-z^2)^* = 1 - z^2$

$$\Leftrightarrow \quad z^{*2} = z^2 \qquad \Rightarrow \quad z = \pm z^*$$

$$\Rightarrow \quad z = x \quad \text{or} \quad z = iy \qquad x, y \in \mathbb{R}$$

for $z = x$ $\quad 1 - x^2 \in \mathbb{R}^-$ for $|x| > 1$ $\Rightarrow$ exclude $(-\infty, -1)$ and $(1, \infty)$

for $z = iy$ $\quad 1 + y^2 \in \mathbb{R}^+$ $\Rightarrow$ no restriction required

ii) Suppose that

$$iz + \exp\left[\frac{1}{2}\log(1-z^2)\right] = r \in \mathbb{R}$$

$$\Leftrightarrow \quad 1 - z^2 = (r - iz)^2$$

$$1 - z^2 = r^2 - 2irz - z^2$$

$$z = i\,\frac{1-r^2}{2r} \notin \mathbb{R}$$

$\Rightarrow$ no restriction imposed from this possibility

$\Rightarrow$ The domain of analyticity of $\arcsin z$ is $\mathbb{C} \setminus \{(-\infty, -1), (1, \infty)\}$

$\boxed{\Sigma = 10}$

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i/ Closure:

$$T \circ T' = \frac{a\left(\frac{a'z+b'}{c'z+d'}\right)+b}{c\left(\frac{a'z+b'}{c'z+d'}\right)+d}$$

$$= \frac{a(a'z+b')+b(c'z+d')}{c(a'z+b')+d(c'z+d')}$$

$$= \left(\frac{(aa'+bc')z+(ab'+bd')}{(ca'+dc')z+(cb'+dd')}\right) = \frac{a''z+b''}{c''z+d''}$$

(4)

Existence of Identity:

$I = T(z) = z$

$$I \circ T(z) = \frac{az+b}{cz+d} = T(z) \circ I$$

(4)

Existence of the Inverse:

$T^{-1}(z) \circ T(z) = T(z) \circ T^{-1}(z) = \cancel{I} = z$

$$\frac{aT^{-1}(z)+b}{cT^{-1}(z)+d} = z$$

$$aT^{-1}(z)+b = czT^{-1}(z)+dz$$

$$T^{-1}(z)(a-cz) = dz-b$$

$$\Rightarrow \quad T^{-1}(z) = \frac{dz-b}{-cz+a}$$

(4)

check $T^{-1}(z) \circ T(z) = z$ ✓

$$\frac{d\left(\frac{az+b}{cz+d}\right)-b}{-c\left(\frac{az+b}{cz+d}\right)+a} = \frac{daz+\cancel{db}-\cancel{bd}-bcz}{-\cancel{caz}-cb+\cancel{dcz}+da} = z$$

associativity:

$$T \circ (T' \circ T'') = (T \circ T') \circ T''$$

$$T \circ \left(\frac{(a'a'' + b'c'')z + (a'b'' + b'd'')}{(c'a'' + d'c'')z + c'b'' + d'd''} \right)$$

(4)

$=$ ✓

closure

$$T \cdot T' = \begin{pmatrix} a a' + b c' & a b' + b d' \\ c a' + d c' & c b' + d d' \end{pmatrix}$$

existence of identity:

T^{-1} exist if $\det T \neq 0$

identity

$T(z)^{-1} \equiv T^{-1}$ $\qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(4)

associativity guaranteed by matrix multiplication

$\boxed{\sum = 20}$