Mathematical Methods II

Coursework 1

Hand in the complete solutions to all three questions in the general office (room C123)

DEADLINE: Thursday 30/10/2008 at 17:00

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1) In the lecture we proved lemma 2, which states that the linear fractional transformation [30 marks]

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad-bc \neq 0; a,b,c,d \in \mathbb{C}$$

always maps circles and lines into circles and lines. The proof was based on the decomposition of $T(z)$ into a succession of translations, rotations and the inversion map together with the fact that these more basic maps respect the stated property.

i) Prove lemma 2 in an alternative way than in the lecture by using the arguments of the proof of lemma 1. Start for this with the parameterization of all possible circles and lines in the z-plane

$$\alpha(x^2+y^2) + \beta x + \gamma y + \delta = 0 \qquad \text{for } \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

and investigate how these configurations are mapped into the image-plane, i.e. the w-plane.

ii) For which values of a, b, c, d and $\alpha, \beta, \gamma, \delta$ are circles mapped into circles, circles mapped into lines, lines mapped into circles and lines mapped into lines?

iii) Recover the inversion map as a special case for $T(z)$ and verify that your answer in ii) reduces to the cases discussed in the lecture.

iv) Use your answer in i) to construct a linear fractional transformation which maps a circle with radius 5 centered at $(2,3)$ into a circle with radius $\sqrt{15}$ centered at $(-1,-3)$.

v) Construct the same map as in iv), but now by use theorem 5, i.e. select three distinct points on the circle in the z-plane, three distinct points on the circle in the w-plane and construct the unique map from these six points.

vi) Construct a linear fractional transformation which maps vertical into horizontal lines.

2) Expressions which do not change when transformed with regard to some symmetry transformation are said to be invariants of that particular symmetry. Accordingly, expressions which do not change when transformed via a linear fractional transformation are said to be $Gl(2,\mathbb{C})$-invariants. [10 marks]

i) Verify that the quantity

$$C(x,t) = \frac{T_t}{T_x}$$

is a $Gl(2,\mathbb{C})$-invariant.

ii) Determine the constant λ, such that

$$S(x,t) = \frac{T_{xxx}}{T_x} + \lambda \left(\frac{T_{xx}}{T_x} \right)^2$$

becomes a $Gl(2,\mathbb{C})$-invariant.

iii) Use the results from i) and ii) to show that

$$\frac{1}{T_t^4} \left(\frac{9}{4} T_{xx}^4 - 3 T_x T_{xx}^2 T_{xxx} + T_x^2 T_{xxx}^2 \right)$$

is a $Gl(2,\mathbb{C})$-invariant.

3) Use Euler's formula to show that [10 marks]

$$\arccos(z) = -i \ln \left(z + \sqrt{z^2 - 1} \right).$$

Subsequently use the principal branch of the logarithmic function to determine the domain of analyticity for $\arccos(z)$.

Mathematical Methods II

Solutions and marking scheme for coursework 1

INSTRUCTIONS: Question 1 carries 30 mark and questions 2 and 3 carry 10 marks each.

1) i) A point $z = x + iy$ on the circle in the z-plane is mapped to a point $w = u + iv$ in the w-plane by

$$w = T(x + iy) = \frac{a(x + iy) + b}{c(x + iy) + d} = u + iv$$

Solving the last equality for x and y gives

$$x = \frac{(a - cu)(du - b) - cdv^2}{(a - cu)^2 + c^2v^2} \quad \text{and} \quad y = \frac{adv - bcv}{(a - cu)^2 + c^2v^2} \tag{1}$$

when $a, b, c, d \in \mathbb{R}$. Alternatively one may use the inverse function

$$T^{-1}(u + iv) = \frac{d(u + iv) - b}{-c(u + iv) + a} = x + iy$$

Separating the real and imaginary part also yields the expressions for x and y in (1). Next we substitute the expressions for x and y from (1) into

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0.$$

After some algebra one finds: 15

$$(\delta c^2 - dc\beta + d^2\alpha)(u^2 + v^2) + (bc\beta + ad\beta - 2bd\alpha - 2ac\delta)u + (ad - bc)\gamma v = ab\beta - b^2\alpha - a^2 d. \tag{2}$$

In the generic case we should take $a, b, c, d \in \mathbb{C}$. Using the convention $a = a_r + ia_i$, $b = b_r + ib_i$, $c = c_r + ic_i$ and $d = d_r + id_i$ with $a_r, a_i, b_r, b_i, c_r, c_i, d_r, d_i$ we obtain

$$x = \frac{b_i(c_iu + c_rv) + b_r(c_ru - c_iv) + a_i(d_iu + d_rv - b_i) - a_r(b_r - d_ru + d_iv)}{D} - \frac{(c_rd_r + c_id_i)(u^2 + v^2)}{D}$$

$$y = \frac{b_i(c_ru - c_iv) - b_r(c_iu + c_rv) + a_r(d_iu + d_rv - b_i) + a_i(b_r - d_ru + d_iv)}{D} + \frac{(c_id_r - c_rd_i)(u^2 + v^2)}{D}$$

where

$$D = a_r^2 + a_i^2 + 2a_r(c_i^2 v - c_r^2 u) - 2a_i(c_i u + c_r v) + \left(c_r^2 + c_i^2\right)\left(u^2 + v^2\right).$$

Clearly this means that (2) becomes more involved.

ii) We have the following possibilities: 2

$$\begin{array}{llll} \text{circles} \to \text{circles:} & \alpha \neq 0 & \wedge & \delta c^2 - dc\beta + d^2\alpha \neq 0 \\ \text{circles} \to \text{lines:} & \alpha \neq 0 & \wedge & \delta c^2 - dc\beta + d^2\alpha = 0 \\ \text{lines} \to \text{circles:} & \alpha = 0 & \wedge & \delta c^2 \neq dc\beta \\ \text{lines} \to \text{lines:} & \alpha = 0 & \wedge & \delta c^2 = dc\beta \end{array} \tag{3}$$

iii) We obtain the inversion map in the limit:

$$\lim_{\substack{a\to 0, b\to 1 \\ c\to 1, d\to 0}} T(z) = f_I(z) = \frac{1}{z}$$

Next we verify that all the equations valid for the linear fractional transformation reduce to those of the inversion map. It is easy to see that equation (2) becomes

$$\delta(u^2 + v^2) + \beta u - \gamma v + \alpha = 0,$$

which is what we found in the lecture.

Futhermore the maps (3) reduce to 3

$$\begin{array}{llll} \text{circles} \to \text{circles:} & \alpha \neq 0 & \wedge & \delta \neq 0 \\ \text{circles} \to \text{lines:} & \alpha \neq 0 & \wedge & \delta = 0 \\ \text{lines} \to \text{circles:} & \alpha = 0 & \wedge & \delta \neq 0 \\ \text{lines} \to \text{lines:} & \alpha = 0 & \wedge & \delta = 0 \end{array} \tag{4}$$

iv) The circle

$$(x-2)^2 + (y-3)^2 = 25 \tag{5}$$

is mapped into the circle

$$(u+1)^2 + (v+3)^2 = 15. \tag{6}$$

From (5) follows

$$(x^2 + y^2) - 4x - 6y - 12 = 0 \qquad \Rightarrow \alpha = 1, \beta = -4, \gamma = -6, \delta = -12 \tag{7}$$

and from (6) follows

$$(u^2 + v^2) + 2u + 6v - 5 = 0. \tag{8}$$

Comparing (2) and (8) yields therefore

$$\begin{array}{ll} 1 = \delta c^2 - dc\beta + d^2\alpha & 6 = (ad - bc)\gamma \\ 2 = bc\beta + ad\beta - 2bd\alpha - 2ac\delta & 5 = ab\beta - b^2\alpha - a^2 d \end{array} \tag{9}$$

These four equations should now be solved for a, b, c, d.There is no real solution to this. $\boxed{4}$

(The question was intended to produce a real solution, but unfortunately there was a typo in the original task sheet. I have given full marks for any good attempt.)

v) We know that fixing three points z_1, z_2 and z_3 and further three points in the image plane w_1, w_2 and w_3, the LFT is uniquely determined by the equation

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}. \tag{10}$$

Let us therefore pick three points on each circle. For instance $z_1 = 7+3i$, $z_2 = 2+8i$, $z_3 = -3+3i$ and $w_1 = \sqrt{15}-1-3i$, $w_2 = -1+(\sqrt{15}-3)i$, $w_3 = -1-\sqrt{15}-3i$. Substituting these points into (10) and solving for w gives

$$w = \sqrt{\frac{3}{5}}z - (1+3i) - (2+3i)\sqrt{\frac{3}{5}}.$$

(Different choices yield different maps. One may verify that these maps are solutions for the maps constructed in i)) $\boxed{4}$

vi) We employ employ again the equation (10), which serves to construct a unique LFT from six point:We take three point from the vertical line, e.g. $z_1 = i$, $z_2 = 0$ $\boxed{2}$ and $z_3 = -i$ and three points from the horizontal line, e.g. $w_1 = -1$, $w_2 = 0$ and $w_3 = 1$. Substituting them into (10) gives

$$\frac{(w+1)(0-1)}{(w-1)(0+1)} = \frac{(z-i)(0+i)}{(z+i)(0-i)}. \tag{11}$$

Solving this for w yields the final answer $\boxed{\sum = 30}$

$$w = iz.$$

2) i) Consider

$$T(f(x,t)) = \frac{af(x,t)+b}{cf(x,t)+d}$$

and compute

$$\frac{d}{dt}T(f(x,t)) = \frac{(ad-bc)f_t}{(d+cf)^2} \quad \text{and} \quad \frac{d}{dx}T(f(x,t)) = \frac{(ad-bc)f_x}{(d+cf)^2}.$$

Therefore

$$C(x,t) = \frac{T_t}{T_x} = \frac{f_t}{f_x}$$

is a $Gl(2,\mathbb{C})$-invariant. $\boxed{1}$

ii) We compute

$$\frac{T_{xxx}}{T_x} = \frac{6c\left(cf_x^2 - (d+cf)f_{xx}\right)}{(d+cf)^2} + \frac{f_{xxx}}{f_x}$$
$$\left(\frac{T_{xx}}{T_x}\right)^2 = \frac{\left((d+cf)f_{xx} - 2cf_x^2\right)^2}{(d+cf)^2 f_x^2}$$

Then

$$S(x,t) = \frac{T_{xxx}}{T_x} + \lambda\left(\frac{T_{xx}}{T_x}\right)^2 = \frac{f_{xxx}}{f_x} + \lambda\left(\frac{f_{xx}}{f_x}\right)^2$$

becomes a $Gl(2,\mathbb{C})$-invariant only for $\lambda = -3/2$. 7

iii) We may re-express the expression in terms of the $Gl(2,\mathbb{C})$-invariants $C(x,t)$ and $S(x,t)$ 2

$$\frac{1}{T_t^4}\left(\frac{9}{4}T_{xx}^4 - 3T_xT_{xx}^2T_{xxx} + T_x^2T_{xxx}^2\right) = \frac{S(x,t)^2}{C(x,t)^4}.$$

Since the right hand side is $Gl(2,\mathbb{C})$-invariant by i) and ii) the left hand side is also $Gl(2,\mathbb{C})$-invariant. $\sum = 10$

3) We use Euler's formula

$$w = \cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(y + y^{-1}) \quad \text{with } y = e^{iz}.$$

Therefore

$$y^2 - wy + 1 = 0$$

which is solved by

$$y_{1/2} = w \pm \sqrt{w^2 - 1}.$$

Therefore selecting one branch of the quare root gives

$$\arccos(z) = -i\ln\left(z + \sqrt{z^2-1}\right).$$

The principal branch of the log has the negative real axis, i.e. $(-\infty, 0) \equiv \mathbb{R}^-$, as branch cut. Writing the right hand side as $-i\ln\left(z + i\sqrt{1-z^2}\right)$ we therefore need to guarantee that

$$i)\ \ 1 - z^2 \notin \mathbb{R}^- \qquad \text{and} \qquad ii)\ \ z + i\exp\left[\frac{1}{2}\ln(1-z^2)\right] \notin \mathbb{R}^-$$

i) Suppose that $1 - z^2 \notin \mathbb{R}$

$$\Rightarrow (1-z^2)^* = 1 - z^2 \quad \Leftrightarrow \quad (z^*)^2 = z^2 \quad \Rightarrow \quad z = \pm z^* \quad \Rightarrow z = x, z = iy$$

for $z = x$: $1 - x^2 \in \mathbb{R}^-$ for $|x| > 1 \Rightarrow$ exclude the intervals $(-\infty, -1)$ and $(1, \infty)$.
for $z = iy$: $1 + y^2 \in \mathbb{R}^+ \Rightarrow$ no further restrictions arise from this possibility.

$ii)$ Assume that

$$z + i \exp\left[\frac{1}{2}\ln(1 - z^2)\right] = r \in \mathbb{R}^-$$

Therefore

$$-(1 - z^2) = (r - z)^2 \quad \Leftrightarrow \quad -1 + z^2 = r^2 + z^2 - 2rz \quad \Rightarrow \; z = \frac{1 + r^2}{r^2}.$$

This means $z \in \mathbb{R}^-$ only for $r \in \mathbb{R}^-$ and no further restriction results from this possibility.

The principle branch cuts of $\cos z$ are therefore $(-\infty, -1)$ and $(1, \infty)$. $\boxed{\sum = 10}$