Mathematical Methods II

Coursework 1

Hand in the complete solutions to all five questions in the general office(room C123).

DEADLINE: Tuesday 3/11/2009 at 16:00

1) Find <u>all</u> values for the expressions [5 marks]

$$z_1 = (1+i)^i \qquad \text{and} \qquad z_2 = 1^{\sqrt{2}},$$

in the form $z = x + iy$ with $x, y \in \mathbb{R}$.

2) Verify that the function [10 marks]

$$w = \frac{\alpha}{2}\left(z + \frac{1}{z}\right) \qquad \text{for } \alpha \in \mathbb{C},$$

maps the exterior of a semicircle with radius one centered at the origin onto the upper half plane.

3) Prove that the most general linear fractional transformation, [10 marks]

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad - bc \neq 0; a, b, c, d \in \mathbb{C}$$

which maps a circle of radius one into a circle of radius one for $a \neq 0$ is given by

$$T(z) = e^{i\theta}\frac{z-\gamma}{\bar{\gamma}z - 1} \qquad \text{for } \theta \in \mathbb{R}, \gamma \in \mathbb{C}.$$

Determine a, b, c, d as functions of θ, γ. Depending on the value of $|\gamma|$ find the region to which the interior of the unit circle, i.e $|z| < 1$, is mapped to by the function you constructed. Find the corresponding expressions for the case $a = 0$.

4) Find a transformation that maps the parabola [5 marks]

$$y = \pm\sqrt{4\alpha(\alpha - x)} \qquad \text{for } \alpha \in \mathbb{R},$$

to a straight line.

5) Find an analytic function that maps the exterior of the unit circle into the interior of a regular hexagon. [20 marks]

Solution CW1 09

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1) $z_1 = (1+i)^i = \exp[\ln(1+i)^i] = \exp[i\ln(1+i)]$

$= \exp[i(\ln\sqrt{2} + i\arctan 1 + 2\pi i n)]$ $\quad \because \ln z = \ln|z| + i\,\mathrm{ARG}\,z + 2\pi i n$, $n \in \mathbb{Z}$

$= \exp\left(\frac{i}{2}\ln 2\right) e^{-\frac{\pi}{4} - 2\pi n}$

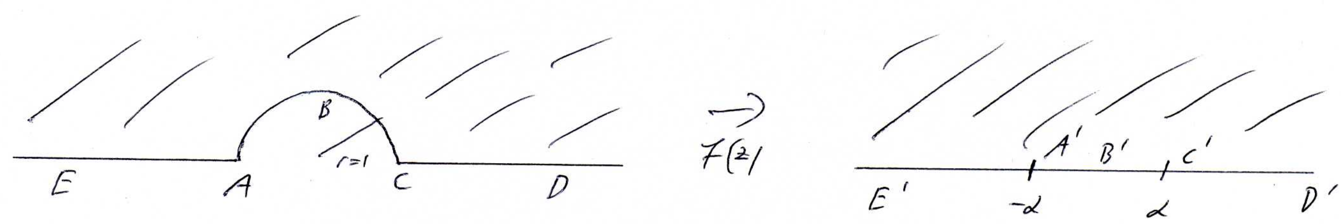
$z_1 = e^{-\frac{\pi}{4}+2\pi n}\cos\left(\frac{1}{2}\ln 2\right) + i\, e^{-\frac{\pi}{2}+2\pi n}\sin\left(\frac{1}{2}\ln 2\right)$ 3

$z_2 = 1^{\sqrt{2}} = \exp(\ln 1^{\sqrt{2}}) = \exp(\sqrt{2}\ln 1) = \exp[\sqrt{2}(\ln 1 + 2\pi i n)]$

$= \exp(2^{\frac{3}{2}} i\pi n)$

$= \cos(2^{\frac{3}{2}} n\pi) + i\sin(2^{\frac{3}{2}} n\pi)$ 2 $\frac{5}{5}$

2)



$f(z) = \frac{\alpha}{2}\left(z + \frac{1}{z}\right)$

First verify that the map is correct for the boundary:

- semicircle $\equiv \widehat{ABC}$: $\quad z = e^{i\theta}$ for $0 \le \theta \le \pi$

$\Rightarrow f(z) = \frac{\alpha}{2}(e^{i\theta} + e^{-i\theta}) = \alpha\cos\theta$

$\Rightarrow$ for $\alpha \in \mathbb{R}$ $f(z)$ maps the semicircle to $[-\alpha, \alpha]$ on the real line

- $(1,\infty) \equiv \overline{CD}$: $\quad z = re^{i\theta}$ with $\theta = 0$, $r \in (1,\infty)$

$\Rightarrow f(z) = \frac{\alpha}{2}\left(r + \frac{1}{r}\right) \in (\alpha, \infty)$ for $r \in (1,\infty)$

- $(-\infty, 1) \equiv \overline{EA}$: $\quad z = re^{i\theta}$ with $\theta = \pi$, $r \in (-\infty, -1)$

$\Rightarrow f(z) = \frac{\alpha}{2}\left(r + \frac{1}{r}\right) \in (-\infty, -\alpha)$ for $r \in (-\infty, 1)$

$\Rightarrow$ The boundaries are mapped correctly. 7

From the Riemann mapping theorem follows that we could have all points from the exterior of the semicircle in the upper half plane mapped either to the upper half plane or to the lower half plane. We can decide which scenario occurs by checking one point. This is enough since the region is simply connected.

Take $z = 2i$:

$$f(2i) = \frac{\alpha}{2}\left(2i + \frac{1}{2i}\right) = \frac{i\alpha}{2}\left(2 - \frac{1}{2}\right) = \frac{i\alpha}{4} \in \text{upper half plane}$$

for $\alpha \in \mathbb{R}^+$

$\Rightarrow$ This proves that the exterior of the semicircle in the upper half plane is mapped to the upper half plane. 3

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3)
$$T(z) = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

$$|T(z)| = 1 = \left|\frac{az+b}{cz+d}\right| \Rightarrow (az+b)(\bar{a}\bar{z}+\bar{b}) = (cz+d)(\bar{c}\bar{z}+\bar{d})$$

$$\Rightarrow \quad a\bar{a}\, z\bar{z} + b\bar{a}\bar{z} + a\bar{b}z + b\bar{b} = c\bar{c}\, z\bar{z} + \bar{d}cz + d\bar{c}\bar{z} + d\bar{d}$$

$\because z\bar{z}$

$$\Rightarrow \quad |a|^2 + |b|^2 + a\bar{b}z + b\bar{a}\bar{z} = |c|^2 + |d|^2 + c\bar{d}z + d\bar{c}\bar{z}$$

$$\Rightarrow \quad |a|^2 + |b|^2 = |c|^2 + |d|^2 \qquad (1)$$

$$a\bar{b} = c\bar{d} \qquad (2)$$

$$b\bar{a} = d\bar{c} \qquad (3)$$

w.e.g. take $d = -1$:

(1): $|a|^2 + |b|^2 = |c|^2 + 1$

(2)·(3): $|a|^2|b|^2 = |c|^2$

$$\Rightarrow |a|^2(1 - |b|^2) = (1 - |b|^2) \Rightarrow |a|^2 = 1$$

$$\Rightarrow \underline{a = e^{i\theta}}$$

$$3 \Rightarrow \quad b e^{-i\theta} = -\bar{c} \quad \Rightarrow \quad \underline{b = -e^{i\theta}\bar{c}}$$

$$\Rightarrow \quad T(z) = \frac{e^{i\theta}z - e^{i\theta}\bar{c}}{cz - 1} = e^{i\theta}\frac{z - \bar{c}}{cz - 1}$$

$$\underline{T(z) = e^{i\theta}\frac{z - \gamma}{\bar{\gamma}z - 1}} \qquad \text{with } \underline{c = \bar{\gamma}}$$

i.e.: $a = e^{i\theta}$, $b = -e^{i\theta}\gamma$, $c = \bar{\gamma}$, $d = -1$ 6

Where is the exterior mapped to?

$$|T(z)|^2 = \left|\frac{z - \gamma}{\bar{\gamma}z - 1}\right|^2 < 1$$

$\Leftrightarrow \quad \dfrac{(z-\gamma)(\bar z-\bar\gamma)}{(\bar\gamma z-1)(\gamma\bar z-1)} = \dfrac{z\bar z-\bar\gamma z-\gamma\bar z+|\gamma|^2}{|\gamma|^2|z|^2-\bar\gamma z-\gamma\bar z+1} < 1$

$\Leftrightarrow \quad |z|^2+|\gamma|^2 < |\gamma|^2|z|^2+1$

$\Leftrightarrow \quad |\gamma|^2(1-|z|^2) < (1-|z|^2)$

$\Rightarrow$ for $|\gamma|<1 \Rightarrow |z|<1$

for $|\gamma|>1 \Rightarrow |z|>1$

$\Rightarrow$ When $|\gamma|<1$ the interior of the unit circle is mapped to the interior of the unit circle and when $|\gamma|>1$ the exterior is mapped to the interior

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Now $a=0$: $\quad T(z) = \dfrac{b}{zc+d}$

$|T(z)|=1 \Rightarrow \quad |b|^2 = (cz+d)(\bar c\bar z+\bar d) = |c|^2|z|^2+c\bar d z+d\bar c\bar z+|d|^2$

$\Rightarrow \quad |b|^2 = |c|^2+|d|^2$

$$\left.\begin{aligned} c\bar d &= 0 \\ d\bar c &= 0 \end{aligned}\right\} \Rightarrow c=0 \quad \text{or } d=0$$

$c=0$ is excluded since $ad-bc\neq 0$

$\Rightarrow \underline{d=0} \Rightarrow |b|^2=|c|^2 \Rightarrow b = e^{i\theta}c$

$\Rightarrow \underline{T(z) = \dfrac{e^{i\theta}c}{cz} = \dfrac{e^{i\theta}}{z} = f_R^{e^{i\theta}} \circ f_I(z)}$

2

$\Rightarrow$ The interior is always mapped to the exterior and vice versa

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4) Suppose a curve in the z-plane is parameterised by:

$x = F(t) \qquad y = G(t)$

$\Rightarrow \quad z = x+iy = F(u+iv) + iG(u+iv)$

$= F(u) + iG(u)$

take $v=0$, i.e. the u-axis becomes the line we are looking for

⇒ The curve is mapped to the real line in the w-plane

A parameterisation for $y^2 = 4\alpha(\alpha - x)$ is

$y = 2\alpha t = G(t)$ and $x = \alpha(1-t^2) = F(t)$

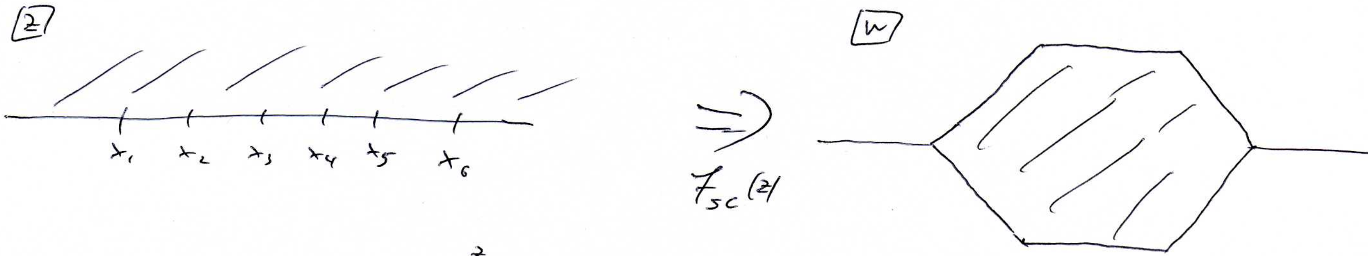
⇒ $z = \alpha - \alpha w^2 + 2\alpha i w$ with $w = u$

⇒ $\underline{w = u = i \pm i\sqrt{\frac{z}{\alpha}}}$ 5

check! $w = i \pm i\sqrt{\frac{x}{\alpha} + i\frac{y}{\alpha}} = i \pm i\sqrt{1-t^2+2\alpha i} = i \pm i(1+it)$

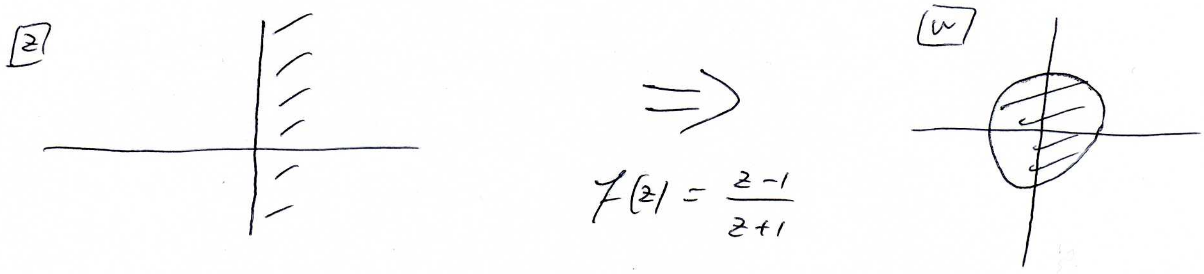
⇒ the "−" sign gives the real axis $\frac{5}{5}$

5) The Schwarz-Christoffel transformation for



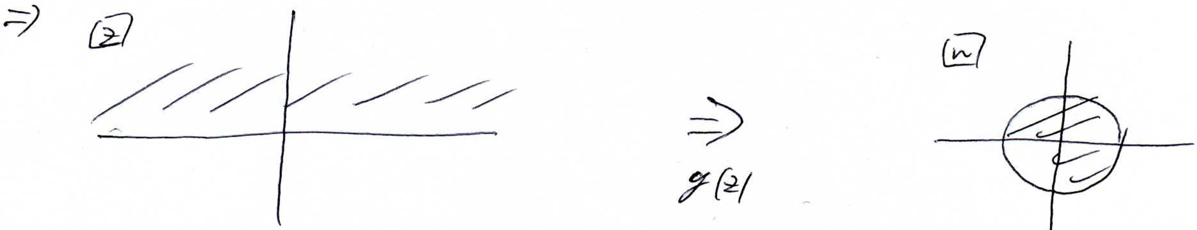
is $f_{SC}(z) = w = c\int^{z} (z'-x_1)^{-\frac{1}{3}}(z'-x_2)^{-\frac{1}{3}}\dots(z'-x_6)^{-\frac{1}{3}}\,dz' + \tilde{c}$

From Ex 3 (Q 2) we know:



$f(z) = \frac{z-1}{z+1}$

⇒

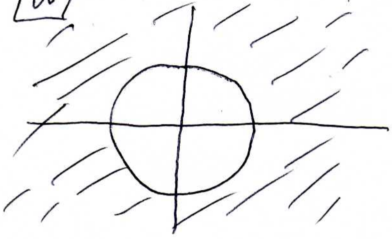


$g(z) = f \circ f_R^{-i = e^{-\frac{i\pi}{2}}}(z) = f(-iz) = \frac{-iz-1}{-iz+1} = \frac{z-i}{z+i}$

⇒

$h(z) = f_I \circ g(z) = \frac{z+i}{z-i}$

$\Rightarrow$ $\boxed{\tilde{w}}$



$\Rightarrow$ $\boxed{z}$

$h^{-1}(\ ) =$

$$h^{-1}(\tilde{w}) = i\,\frac{\tilde{w}+1}{\tilde{w}-1} = z$$

$$\Rightarrow \frac{dz}{d\tilde{w}} = -\frac{2i}{(\tilde{w}-1)^2}$$

$$\Rightarrow (z - x_i) = i\left(\frac{1+\tilde{w}}{\tilde{w}-1}\right) - i\,\frac{\tilde{w}_i+1}{\tilde{w}_i-1} = 2i\,\frac{\tilde{w}_i-\tilde{w}}{(\tilde{w}-1)(\tilde{w}_i-1)}$$

$$\Rightarrow w = c\int^{h(z)} \prod_{i=1}^{6}\left(\frac{2i(\tilde{w}_i-\tilde{w})}{(\tilde{w}-1)(\tilde{w}_i-1)}\right)^{-\frac{1}{3}} \frac{-2i}{(\tilde{w}-1)^2}\,d\tilde{w} + \bar{c}$$

$$= \underbrace{c\,\frac{-2i}{(2i)^2}\prod_{i=1}^{6}(\tilde{w}_i-1)^{\frac{1}{3}}}_{c'} \int^{h(z)} \frac{\prod_{i=1}^{6}(\tilde{w}_i-\tilde{w})^{-\frac{1}{3}}\,\cancel{(\tilde{w}-1)^2}\,d\tilde{w}}{\cancel{(\tilde{w}-1)^2}} +$$

$$w = c'\int^{h(z)} \prod_{i=1}^{6}(\tilde{w}_i-\tilde{w})^{-\frac{1}{3}}\,d\tilde{w} + \bar{c}$$

$\frac{20}{20}$

$\boxed{\Sigma = 50}$