Mathematical Methods II

Coursework 1

Hand in the complete solutions to all five questions to the SEMS general office (C109).

DEADLINE: Friday 25/02/2011 at 12:00

1) Determine the constant λ such that the function [8 marks]

$$u(x,y) = x^4 + \lambda x^2 y^2 + y^4$$

becomes a harmonic function. Compute its conjugate harmonic function $v(x,y)$ and thereafter construct an analytic function using $u(x,y)$ and $v(x,y)$.

2) In the definition of the general linear fractional transformation [2 marks]

$$w = T(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}$$

one assumes $ad - bc \neq 0$. Provide a reason why this constraint is needed.

3) Construct a conformal transformation that maps a circle centered at $z = 2 + 2i$ with radius $r = 2$ to the line passing through the points $w = i$ and $w = -1$. [10 marks]

4) i) For the line segment $\mathcal{L}_z$ and the semi-circle $\mathcal{C}_z$ [10 marks]

$$\mathcal{L}_z = \{x, y : x = \ln r,\ 0 < y < \pi\} \qquad \mathcal{C}_z = \{r, \theta : r \in \mathbb{R}^+, 0 < \theta < \pi\},$$

show that the function

$$f_1(z) = e^z$$

maps $\mathcal{L}_z$ onto $\mathcal{C}_z$.

ii) Determine the length of the major and minor axis of the ellipse onto which the function

$$f_2(z) = z + \frac{1}{z}$$

maps the semi-circle $\mathcal{C}_z$.

iii) Construct a conformal map that maps the line segment $\mathcal{L}'_z = \{x, y : x = \pi/4,\ 0 < y < \pi\}$ onto an ellipse centered at the origin with major axis length $a = 4\cosh \pi/4$ and minor axis length $b = 4 \sinh \pi/4$.

[20 marks]

5) i) Find a domain on which the function

$$g_1(z) = \ln\left(\frac{z-4}{z^2-4}\right)$$

is single valued and analytic. Provide two alternative constructions: a) Take the principal branch cut for $\ln(z)$ and b) take the branch cut for $\ln(z)$ to be $\mathbb{R}^+$.

ii) Find a domain on which the function

$$g_2(z) = \operatorname{arcsinh} z$$

is single valued and analytic. Use the principal branch cut for $\ln(z)$.

Mathematical Methods II

Coursework 1

Solutions and marking scheme

DEADLINE: Friday 25/02/2011 at 12:00

1) Compute [8 marks]

$$\frac{d^2u}{dx^2} = 2(6x^2 + \lambda y^2) \quad \text{and} \quad \frac{d^2u}{dy^2} = 2(6y^2 + \lambda x^2)$$

Therefore

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 2(x^2 + y^2)(6 + \lambda),$$

such that $u(x, y)$ becomes a harmonic function when [3]

$$\boxed{\lambda = -6}.$$

Using the Cauchy Riemann equations

$$\begin{aligned} \partial_x u(x, y) &= 4x^3 - 12xy^2 = \partial_y v(x, y) \\ \partial_y u(x, y) &= -12x^2 y + 4y^3 = -\partial_x v(x, y) \end{aligned}$$

gives after integration

$$\begin{aligned} v(x, y) &= 4x^3 (y + f_1(x)) - 12x \left(\frac{y^3}{3} + f_2(x)\right) = 4yx^3 + 4f_1(x)x^3 - 4y^3 x - 12f_2(x)x \\ v(x, y) &= 12y \left(\frac{x^3}{3} + g_1(y)\right) - 4y^3 (x + g_2(y)) = 4yx^3 - 4y^3 x + 12yg_1(y) - 4y^3 g_2(y) \end{aligned}$$

Comparing these two equations yields the conjugate harmonic function of $u(x, y)$ [4]

$$\boxed{v(x, y) = 4x^3 y - 4xy^3}.$$

This means the function [1]

$$\begin{aligned} f(x, y) &= u(x, y) + iv(x, y) \\ &= \boxed{x^4 - 6x^2 y^2 + y^4 + i(4x^3 y - 4xy^3)} \end{aligned}$$

is analytic.

2) The restriction is needed as otherwise $T'(z) = \frac{ad-bc}{(d+cz)^2} = 0$, i.e. the map would just be a constant. [2 marks]

3) We constuct a linear fractional transformation by selecting three points z_1, z_2, z_3 on the circle and three points on the line w_1, w_2, w_3 [10 marks]

$$\begin{aligned} z_1 &= 2i, \quad z_2 = 2, \quad z_3 = 4 + 2i \\ w_1 &= -1, \quad w_2 = i, \quad w_3 = 1 + 2i. \end{aligned}$$

(Different choices for the points are possible.) Substituting this into

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

and solving for w gives the linear fractional transformation

$$\boxed{w = \tfrac{z+2}{z-(2+4i)}}.$$

4) i) Substituting $z = \ln r + iy$ with $0 < y < \pi$ into $f_1(z)$ [10 marks]

$$f_1(z) = e^z = e^{\ln r + iy} = re^{iy}$$

gives precisely the semicircle when $\mathcal{C}_z = \{r, \theta : r \in \mathbb{R}^+, 0 < \theta < \pi\}$. $\boxed{1}$

ii) We have $\boxed{6}$

$$\begin{aligned} w &= f_2(z) = z + \frac{1}{z} = u + iv = \frac{z^2+1}{z} = \frac{z^2\bar{z} + \bar{z}}{z\bar{z}} \\ &= \frac{(x+iy)(x^2+y^2) + (x-iy)}{(x^2+y^2)} \\ &= \left(1 + \frac{1}{r^2}\right)x + i\left(1 - \frac{1}{r^2}\right)y \quad \text{with } x^2 + y^2 = r^2. \end{aligned}$$

Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ gives

$$\begin{aligned} u(x,y) &= \left(1 + \frac{1}{r^2}\right)x = \left(r + \frac{1}{r}\right)\cos\theta \\ v(x,y) &= \left(1 - \frac{1}{r^2}\right)y = \left(r - \frac{1}{r}\right)\sin\theta. \end{aligned}$$

Therefore

$$x(u,v) = \frac{u}{(1 + \frac{1}{r^2})} \quad \text{and} \quad y(u,v) = \frac{v}{(1 - \frac{1}{r^2})}.$$

This means

$$r^2 = x^2 + y^2 = \frac{u^2}{\left(1 + \frac{1}{r^2}\right)^2} + \frac{v^2}{\left(1 - \frac{1}{r^2}\right)^2}$$

such that the equation for the ellipse in normal form becomes

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1.$$

Therefore

$$\boxed{\text{length of the major axis} = 2\left(r + \tfrac{1}{r}\right)}$$
$$\boxed{\text{length of the minor axis} = 2\left(r - \tfrac{1}{r}\right)}.$$

iii) We have $f_1 : \mathcal{L}_z \to \mathcal{C}_z$ and $f_2 : \mathcal{C}_z \to$ ellipse. Therefore 3

$$f(z) = f_2 \circ f_1(z) = 2\left(e^z + e^{-z}\right) = 2\cosh z$$

maps $\mathcal{L}'_z = \{x, y : x = \pi/4,\ 0 < y < \pi\}$ onto an ellipse centered when $\ln r = \pi/4$. With the result from ii)

$$\text{length of the major axis} = 2\left(e^{\pi/4} + e^{-\pi/4}\right) = 4\cosh\frac{\pi}{4}$$
$$\text{length of the minor axis} = 2\left(e^{\pi/4} - e^{-\pi/4}\right) = 4\sinh\frac{\pi}{4}.$$

5) (i) The function $g_1(z)$ has three branch points at $z = 4$ and at $z = \pm 2$. For the arguments of the logarithm we can write [20 marks] 10

$$z - 4 = |z-4|\, e^{i\theta_1} \qquad \text{and} \qquad z \pm 2 = |z \pm 2|\, e^{i\theta_{2/3}}$$

such that

$$g_1(z) = \ln\left(\frac{z-4}{z^2-4}\right) = \ln(z-4) - \ln(z-2) - \ln(z+2) = \ln\left|\frac{z-4}{z^2-4}\right| + i(\theta_1 - \theta_2 - \theta_3)$$

We have now various choices for the restriction on θ_1, θ_2 and θ_3 :

a) Assume the principal values for the logarithms:

$$-\pi < \theta_1, \theta_2, \theta_3 \leq \pi$$

Let us now consider the different regions on the real axis:

- $z \in (4, \infty)$: On this part of the axis there is no problem as θ_1, θ_2 and θ_3 are all continuous when crossing the axis.
- $z \in (2, 4)$: On this line segment θ_3 and θ_2 are continuous, but θ_1 jumps and therefore we require a cut.
- $z \in (-2, 2)$: When crossing this part of the axis both θ_1 and θ_2 are discontinuous. However, the relevant quantity, which is the difference $\theta_1 - \theta_2 - \theta_3$ is continuous. Above the axis we have $\theta_3 = 0$, $\theta_1 = \theta_2 = \pi$, such that $\theta_1 - \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_3 = 0$, $\theta_1 = \theta_2 = -\pi$ and therefore also $\theta_1 - \theta_2 - \theta_3 = 0$. This means no cut is required on this segment.
- $z \in (-\infty, -2)$: On this line segment we have above the axis $\theta_1 = \theta_2 = \theta_3 = \pi$ such that $\theta_1 - \theta_2 - \theta_3 = -\pi$ and below the axis we have $\theta_1 = \theta_2 = \theta_3 = -\pi$ such that $\theta_1 - \theta_2 - \theta_3 = \pi$. This means the function is discontinuous and we need a branch cut to make it analytic.

Overall we only need therefore branch cuts at the line segment $(-\infty, -2)$ and $(2, 4)$ in order to make the function $g_1(z)$ single valued and analytic.

b) Next we assume the cut for the logarithms to be at:

$$0 < \theta_1, \theta_2, \theta_3 \leq 2\pi$$

Again we consider the different regions on the real axis:

- $z \in (4, \infty)$: On this line segment we have above the axis $\theta_1 = \theta_2 = \theta_3 = 0$ such that $\theta_1 - \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_1 = \theta_2 = \theta_3 = 2\pi$ such that $\theta_1 - \theta_2 - \theta_3 = -2\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (2, 4)$: On this line segment we have above the axis $\theta_1 = \pi$, $\theta_3 = \theta_2 = 0$, such that $\theta_1 - \theta_2 - \theta_3 = \pi$ and below the axis we have $\theta_1 = \pi$, $\theta_2 = \theta_3 = 2\pi$ and therefore also $\theta_1 - \theta_2 - \theta_3 = -3\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (-2, 2)$: On this line segment we have above the axis $\theta_3 = 0$, $\theta_1 = \theta_2 = \pi$, such that $\theta_1 - \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_3 = 2\pi$, $\theta_1 = \theta_2 = \pi$ and therefore also $\theta_1 - \theta_2 - \theta_3 = -2\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (-\infty, -2)$: On this part of the axis there is no problem as θ_1, θ_2 and θ_3 are all continuous when crossing the axis.

Overall we need therefore a branch cut at the line segment $(-2, \infty)$ in order to make the function $g_1(z)$ single valued and analytic.

(ii) First express the arcsinh in terms of ln 10

$$w = \sinh z = \frac{1}{2}(e^z - e^{-z}) = \frac{1}{2}(y - y^{-1}) \quad \text{with } y = e^z.$$

Therefore

$$y^2 - 2wy - 1 = 0$$

which is solved by

$$y_{1/2} = w \pm \sqrt{w^2 + 1}.$$

Therefore taking the positive square root

$$\text{arcsinh}(z) = \ln\left(z + \sqrt{z^2 + 1}\right)$$

The principal branch of ln has the negative real axis, i.e. $(-\infty, 0) \equiv \mathbb{R}^-$, as branch cut. Thus we need to guarantee that

$$a) \;\; z^2 + 1 \notin \mathbb{R}^- \qquad \text{and} \qquad b) \;\; z + \exp\left[\frac{1}{2}\ln(z^2 + 1)\right] \notin \mathbb{R}^-$$

a) Suppose that $z^2 + 1 \in \mathbb{R}$

$$\Rightarrow (z^2 + 1)^* = z^2 + 1 \quad \Leftrightarrow \quad (z^*)^2 = z^2 \quad \Rightarrow \quad z = \pm z^* \quad \Rightarrow z = x, z = iy$$

for $z = x$: $x^2 + 1 \in \mathbb{R}^+ \Rightarrow$ no restrictions arises from this possibility.
for $z = iy$: $-y^2 + 1 \in \mathbb{R}^-$ for $|y| > 1 \Rightarrow$ we need to cut out $\{(-i\infty, -i), (i, i\infty)\}$.

b) Assume that

$$z + \exp\left[\frac{1}{2}\ln(z^2 + 1)\right] = r \in \mathbb{R}^-$$

Therefore

$$(1+z^2) = (r-z)^2 \quad \Leftrightarrow \quad 1+z^2 = r^2 + z^2 - 2rz \quad \Rightarrow \; z = \frac{r^2-1}{2r}.$$

This means $z \in \mathbb{R}^-$ only for $r \in \mathbb{R}^-$ and no further restriction results from this possibility.

The principal branch cuts of $\operatorname{arcsinh}(z)$ are therefore at $(-i\infty, -i)$ and $(i, i\infty)$.