Mathematical Methods II

Coursework 1

Hand in the complete solutions to all five questions to the SEMS general office (C109).

DEADLINE: Monday 12/03/2012 at 13:00

1) **i)** Given a general linear fractional transformation [10 marks]

$$w = T(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}$$

with $ad - bc \neq 0$, derive the formula

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)},$$

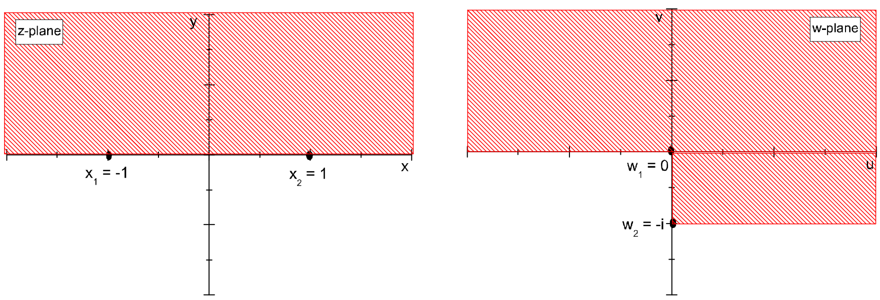
that defines the map which maps three distinct points z_1, z_2, z_3 uniquely into three distinct points w_1, w_2, w_3.

ii) Find the linear fractional transformation $T(z)$ that maps the three points $z_1 = 0$, $z_2 = i$, $z_3 = -1$ uniquely into the three points $w_1 = i$, $w_2 = -1$, $w_3 = 0$.

2) Use the Schwarz-Christoffel transformation to construct an analytic function that maps the upper half plane $\operatorname{Im} z > 0$ into the polygonial region [10 marks]

$$\mathcal{D} = \{u, v : u < 0, v > 0; u \geq 0, v > -i\}$$

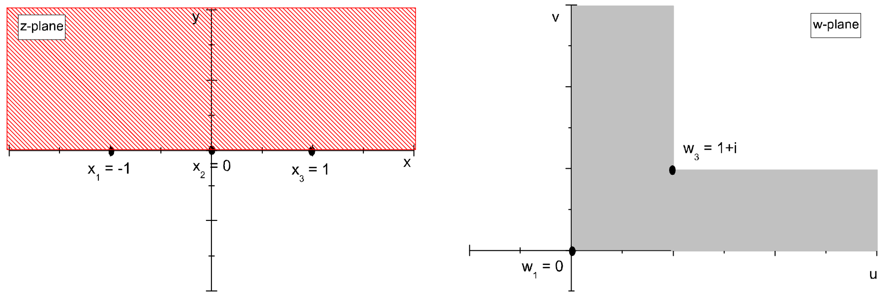
as depicted in the figure:



Hint: Choose $x_1 = -1$, $x_2 = 1$, $w_1 = 0$ and $w_2 = -i$.

3) Determine the equation of the curve in the w-plane which is obtained when the line $x + y = 1$ in the z-plane is transformed with *i)* $w = z^2$ and *ii)* $w = 1/z$. [5 marks]

4) Use the Schwarz-Christoffel transformation to construct an analytic function that maps the upper half plane $\text{Im}\, z > 0$ into the first quadrant bounded by the coordinate axis and the rays $x \geq 0$, $y = 1$ and $y \geq 1$, $x = 1$ as indicated in the figure: [15 marks]



Hint: Choose $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $w_1 = 0$, $w_2 = \alpha$ and $w_3 = 1 + i$. Then take the limit $\alpha \to \infty$ in order to obtain the desired region in the w-plane.

5) Find a domain on which the function [10 marks]

$$g(z) = \ln\left(\frac{z^2 - 9}{z - 1}\right)$$

is single valued and analytic. Provide two alternative constructions: i) Take the principal branch cut for $\ln(z)$ and ii) take the branch cut for $\ln(z)$ to be $\mathbb{R}^+$.

Mathematical Methods II

Solutions coursework 1

Hand in the complete solutions to all five questions to the SEMS general office (C109).

DEADLINE: Monday 12/03/2012 at 13:00

1) i) Compute [10 marks]

$$w - w_i = \frac{az+b}{cz+d} - \frac{az_i+b}{cz_i+d} = \frac{(ad-bc)\,(z-z_i)}{(d+cz)\,(d+cz_i)},$$

such that

$$w - w_1 = \frac{(ad-bc)\,(z-z_1)}{(d+cz)\,(d+cz_1)}, \quad w - w_3 = \frac{(ad-bc)\,(z-z_3)}{(d+cz)\,(d+cz_3)},$$

$$w_2 - w_1 = \frac{(ad-bc)\,(z_2-z_1)}{(d+cz_2)\,(d+cz_1)}, \quad w_2 - w_3 = \frac{(ad-bc)\,(z_2-z_3)}{(d+cz_2)\,(d+cz_3)}.$$

Since $ad - bc \neq 0$ we obtain

$$\begin{aligned} & \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} \\ = & \frac{\cancel{(ad-bc)}\,(z-z_1)}{(d+cz)\,(d+cz_1)} \frac{\cancel{(ad-bc)}\,(z_2-z_3)}{(d+cz_2)\,(d+cz_3)} \frac{(d+cz)\,(d+cz_3)}{\cancel{(ad-bc)}\,(z-z_3)} \frac{(d+cz_2)\,(d+cz_1)}{\cancel{(ad-bc)}\,(z_2-z_1)} \\ = & \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}. \end{aligned}$$

[5]

ii) Substituting the three points $z_1 = 0$, $z_2 = i$, $z_3 = -1$ and $w_1 = i$, $w_2 = -1$, $w_3 = 0$ into the formula yields

$$\frac{(w-i)(-1-0)}{(w-0)(-1-i)} = \frac{(z-0)(i--1)}{(z--1)(i-0)}.$$

Solving this equation for w leads to the linear fractional transformation

$$T(z) = \frac{iz+i}{1-z}.$$

[5]

2) First we identify the angles needed in the Schwarz-Christoffel transformation. Tracing along the boundary from the left to the right the vector is first turned by $-\pi/2$ and then by $\pi/2$. Therefore we have $\mu_1 = -1/2$ and $\mu_2 = 1/2$, such that [10 marks]

$$f'(z) = c(z+1)^{1/2}(z+1)^{1/2} = c\frac{1+z}{\sqrt{z^2-1}}.$$

Integrating we find

$$\begin{aligned} f(z) &= c\int \frac{1}{\sqrt{z^2-1}}dz + c\int \frac{z}{\sqrt{z^2-1}}dz \\ &= c\ln\left[z+\sqrt{z^2-1}\right] + c\sqrt{z^2-1} + \tilde{c} \\ &= c\sqrt{z^2-1} - ic\arcsin z + \tilde{c}. \end{aligned}$$

Next we fix the constants. We have

$$\begin{aligned} f(-1) &= 0 - ic\arcsin(-1) + \tilde{c} = i\frac{c\pi}{2} + \tilde{c} = 0, \\ f(1) &= 0 - ic\arcsin(1) + \tilde{c} = -i\frac{c\pi}{2} + \tilde{c} = -i, \end{aligned}$$

such that

$$\tilde{c} = -\frac{i}{2} \quad \text{and} \quad c = \frac{1}{\pi}.$$

This means

$$f(z) = \frac{1}{\pi}\sqrt{z^2-1} - \frac{i}{\pi}\arcsin z - \frac{i}{2}.$$

3) *i)* We compute [5 marks]

$$w = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy \Rightarrow u = x^2 - y^2, \quad v = 2xy,$$

with $y = 1 - x$ follows

$$\begin{aligned} u &= x^2 - (1-x)^2 = 2x - 1. \\ v &= 2x - 2x^2. \end{aligned}$$

Eliminating x gives

$$v = 1 + u - \frac{2(1+u)^2}{4} = \frac{1}{2}(1-u^2) \;\text{ or }\; u^2 + 2v = 1$$

ii) We compute

$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \Rightarrow u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$

Solving these equation for x and y gives

$$x = \frac{u}{u^2+v^2}, \quad \text{and} \quad y = \frac{-v}{u^2+v^2},$$

such that

$$x + y = 1 \qquad \Rightarrow \qquad u^2 + v^2 = u - v.$$

4) First we identify the angles needed in the Schwarz-Christoffel transformation. Tracing along the boundary from the top to the right the vector is first turned by $\pi/2$, then by ϕ and finally by $\pi/2 - \phi$. As α tends to infinity the angles become [15 marks]

$$\theta_1 = \frac{\pi}{2}, \qquad \theta_2 = \pi, \quad \text{and} \quad \theta_3 = -\frac{\pi}{2}.$$

Therefore we have $\mu_1 = 1/2$, $\mu_2 = 1$ and $\mu_3 = -1/2$, such that

$$\begin{aligned} f'(z) &= c(z+1)^{-1/2} z^{-1} (z-1)^{1/2} = c\frac{1}{z}\sqrt{\frac{z-1}{1+z}} \\ &= c\frac{1}{z}\frac{z-1}{\sqrt{z^2-1}} \end{aligned}$$

Integrating we find

$$\begin{aligned} f(z) &= c\int \frac{1}{\sqrt{z^2-1}}dz - c\int \frac{1}{z\sqrt{z^2-1}}dz \\ &= -ic\int \frac{1}{\sqrt{1-z^2}}dz - c\int \frac{1}{z\sqrt{z^2-1}}dz \\ &= -ic\arcsin z + c\arcsin 1/z + \tilde{c}. \end{aligned}$$

Next we fix the constants. We have

$$\begin{aligned} f(-1) &= i\frac{c\pi}{2} - \frac{c\pi}{2} + \tilde{c} = 0, \\ f(1) &= -i\frac{c\pi}{2} + \frac{c\pi}{2} + \tilde{c} = 1 + i, \end{aligned}$$

such that

$$\tilde{c} = \frac{i+1}{2} \quad \text{and} \quad c = \frac{i}{\pi}.$$

This means

$$f(z) = \frac{1}{\pi}\arcsin z + \frac{i}{\pi}\arcsin 1/z + \frac{i+1}{2}.$$

5) The function $g(z)$ has three branch points at $z = 1$ and at $z = \pm 3$. For the arguments of the logarithm we can write [10 marks]

$$z \pm 3 = |z \pm 3|\, e^{i\theta_{1/2}} \qquad \text{and} \qquad z - 1 = |z-1|\, e^{i\theta_3}$$

such that

$$g(z) = \ln\left(\frac{z^2-9}{z-1}\right) = \ln(z+3) + \ln(z-3) - \ln(z-1) = \ln\left|\frac{z^2-9}{z-1}\right| + i(\theta_1 + \theta_2 - \theta_3)$$

We have now various choices for the restriction on θ_1, θ_2 and θ_3 :

i) Assume the principal values for the logarithms: [5]

$$-\pi < \theta_1, \theta_2, \theta_3 \leq \pi$$

Let us now consider the different regions on the real axis:

- $z \in (3, \infty)$: On this part of the axis there is no problem as θ_1, θ_2 and θ_3 are all continuous when crossing the axis.
- $z \in (1, 3)$: On this line segment θ_2 and θ_3 are continuous, but θ_1 jumps and therefore we require a cut.
- $z \in (-3, 1)$: When crossing this part of the axis both θ_1 and θ_3 are discontinuous. However, the relevant quantity, which is the difference $\theta_1 + \theta_2 - \theta_3$ is continuous. Above the axis we have $\theta_2 = 0$, $\theta_1 = \theta_3 = \pi$, such that $\theta_1 + \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_2 = 0$, $\theta_1 = \theta_3 = -\pi$ and therefore also $\theta_1 + \theta_2 - \theta_3 = 0$. This means no cut is required on this segment.
- $z \in (-\infty, -3)$: On this line segment we have above the axis $\theta_1 = \theta_2 = \theta_3 = \pi$ such that $\theta_1 + \theta_2 - \theta_3 = \pi$ and below the axis we have $\theta_1 = \theta_2 = \theta_3 = -\pi$ such that $\theta_1 + \theta_2 - \theta_3 = -\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.

Overall we only need therefore branch cuts at the line segment $(-\infty, -3)$ and $(1, 3)$ in order to make the function $g(z)$ single valued and analytic.

ii) Next we assume the cuts for the logarithms to be at: [5]

$$0 < \theta_1, \theta_2, \theta_3 \leq 2\pi$$

Again we consider the different regions on the real axis:

- $z \in (3, \infty)$: On this line segment we have above the axis $\theta_1 = \theta_2 = \theta_3 = 0$ such that $\theta_1 + \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_1 = \theta_2 = \theta_3 = 2\pi$ such that $\theta_1 + \theta_2 - \theta_3 = 2\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (1, 3)$: On this line segment we have above the axis $\theta_1 = \pi$, $\theta_3 = \theta_2 = 0$, such that $\theta_1 + \theta_2 - \theta_3 = \pi$ and below the axis we have $\theta_1 = \pi$, $\theta_2 = \theta_3 = 2\pi$ and therefore also $\theta_1 + \theta_2 - \theta_3 = \pi$. This means the function is continuous and we do not need a branch cut to make it analytic.
- $z \in (-3, 1)$: On this line segment we have above the axis $\theta_2 = 0$, $\theta_1 = \theta_3 = \pi$, such that $\theta_1 + \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_2 = 2\pi$, $\theta_1 = \theta_3 = \pi$ and therefore we have $\theta_1 + \theta_2 - \theta_3 = 2\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (-\infty, -3)$: On this part of the axis there is no problem as θ_1, θ_2 and θ_3 are all continuous when crossing the axis.

Overall we need therefore a branch cut at the line segment $(-3, 1)$ and $(3, \infty)$ in order to make the function $g(z)$ single valued and analytic.