Mathematical Methods II

Coursework 1

Hand in the complete solutions to all five questions to the SEMS general office (C109).

DEADLINE: Tuesday 12/03/2013 at 16:00

1) [15 marks] Prove that the most general linear fractional transformation,

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad - bc \neq 0; a, b, c, d \in \mathbb{C}$$

which maps a circle of radius one into a circle of radius one for $a \neq 0$ is given by

$$T(z) = e^{i\theta} \frac{z - \gamma}{\bar{\gamma} z - 1} \qquad \text{for } \theta \in \mathbb{R}, \gamma \in \mathbb{C}.$$

Determine a, b, c, d as functions of θ, γ. Fix θ and γ in such a way that $T(z)$ leaves the unit circle invariant and maps the line passing through the points $z_1 = 0$, $z_2 = (\sqrt{3} + i)/2$ to the line passing through $w_1 = 2(\sqrt{3} + i)$, $w_2 = (-4\sqrt{3} + i11)/13$.

2) [5 marks] Use the definition *The function $f(z)$ is said to possess the limit w_0 as z tends to z_0*, i.e. $\lim_{z \to z_0} f(z) = w_0$, *iff for every $\epsilon > 0$ there exists a $\delta > 0$, such that $|f(z) - w_0| < \epsilon$ for all values of z for which $|z - z_0| < \delta, z \neq z_0$* to prove that

$$\lim_{z \to 1+i} (2 + i)z = 1 + 3i.$$

3) [10 marks] Verify that the functions

$$f(x, y) = e^{x^2 - y^2} \cos(2xy) \qquad \text{and} \qquad g(x, y) = \arctan\left(-\frac{y}{x}\right)$$

are harmonic functions. Compute for each function their conjugate harmonic function and subsequently construct two analytic functions with real parts $f(x, y)$ and $g(x, y)$, respectively.

4) [10 marks] Determine the image of the semi-unit disk in the upper half plane when mapped by the function

$$p(z) = e^{-i\pi/2} \frac{z^2 + 2iz + 1}{z^2 - 2iz + 1}.$$

5) [10 marks] Find a domain on which the function

$$q(z) = z\sqrt{1 - 1/z}$$

is single valued and analytic.

Mathematical Methods II

Solutions for coursework 1

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DEADLINE: Friday 15/03/2013 at 16:00, returned Monday 25/03/2013

1) For the unit circle in the image plane we have [8]

$$|T(z)| = \left| \frac{az+b}{cz+d} \right| = 1,$$

which means

$$|az+b| = |cz+d| \Rightarrow \quad (az+b)(\bar{a}\bar{z}+\bar{b}) = (\bar{c}\bar{z}+\bar{d})(cz+d).$$

Therefore

$$a\bar{a}z\bar{z} + a\bar{b}z + \bar{a}b\bar{z} + b\bar{b} = c\bar{c}z\bar{z} + c\bar{d}z + \bar{c}d\bar{z} + d\bar{d}.$$

Using the fact that $z\bar{z} = 1$ this becomes

$$|a|^2 + |b|^2 + a\bar{b}z + \bar{a}b\bar{z} = |c|^2 + |d|^2 + c\bar{d}z + \bar{c}d\bar{z}$$

Comparing coefficients yields

$$|a|^2 + |b|^2 = |c|^2 + |d|^2, \qquad a\bar{b} = c\bar{d}, \quad \text{and} \quad \bar{a}b = \bar{c}d.$$

Since the constants a, b, c and d are only fixed up to an overall constant we can choose $\underline{d = -1}$ without loss of generality. Multiplying also the last two equations gives

$$\left.\begin{array}{c} |a|^2 + |b|^2 = |c|^2 + 1 \\ |a|^2 |b|^2 = |c|^2 \end{array}\right\} \Rightarrow |a|^2 \left(1 - |b|^2\right) = \left(1 - |b|^2\right) \ \Rightarrow |a|^2 = 1 \ \Rightarrow \underline{a = e^{i\theta}}$$

Therefore

$$\bar{a}b = \bar{c}d \ \Rightarrow \ be^{-i\theta} = -\bar{c} \quad \Rightarrow \ b = -\bar{c}e^{i\theta} = -\gamma e^{i\theta} \quad \text{with } \bar{c} = \gamma,$$

such that

$$T(z) = \frac{e^{i\theta}z - e^{i\theta}\gamma}{\bar{\gamma}z - 1} = e^{i\theta}\frac{z-\gamma}{\bar{\gamma}z - 1}.$$

$T(z)$ leaves the unit circle invariant. Next we fix θ and γ such it also maps the [7] line passing through the points $z_1 = 0$, $z_2 = (\sqrt{3} + i)/2$ to the line passing through $w_1 = 2(\sqrt{3} + i)$, $w_2 = (-4\sqrt{3} + i11)/13$. We have the constraints

$$T(z_1) = w_1 \qquad \text{and} \qquad T(z_2) = w_2$$

Parameterizing $\gamma = re^{i\phi}$ and noting that $w_1 = 2(\sqrt{3}+i) = 4e^{i\pi/6}$ the first constraint becomes

$$e^{i\theta}\gamma = re^{i(\theta+\phi)} = 2(\sqrt{3} + i) = 4e^{i\pi/6} \quad \Rightarrow \; r = 4, \theta + \phi = \frac{\pi}{6}.$$

The second constraint gives

$$T(z_2) = T(e^{i\pi/6}) = e^{i(\pi/6-\phi)} \frac{e^{i\pi/6} - 4e^{i\phi}}{4e^{-i\phi}e^{i\pi/6} - 1} = \frac{e^{i\pi/3}e^{-i\phi} - 4e^{i\pi/6}}{4e^{-i\phi}e^{i\pi/6} - 1} = w_2$$

Solving this for $e^{-i\phi}$ and subsequent simplification gives

$$\begin{aligned} e^{-i\phi} &= \frac{4e^{i\pi/6} - w_2}{e^{i\pi/3} - 4e^{i\pi/6}w_2} = \frac{2(\sqrt{3}+i) - (-4\sqrt{3} + i11)/13}{1/2 + i\sqrt{3}/2 - 2(\sqrt{3}+i)(-4\sqrt{3}+i11)/13} \\ &= \frac{\frac{30}{13}\sqrt{3} + \frac{15}{13}i}{\frac{105}{26} - \frac{15}{26}i\sqrt{3}} = \frac{60\sqrt{3} + 30i}{105 - 15i\sqrt{3}} = \frac{4\sqrt{3} + 2i}{7 - i\sqrt{3}} = \frac{(4\sqrt{3}+2i)(7+i\sqrt{3})}{(7-i\sqrt{3})(7+i\sqrt{3})} \\ &= \frac{26\sqrt{3} + 26i}{52} = \frac{1}{2}\sqrt{3} + \frac{1}{2}i = e^{i\pi/6}. \end{aligned}$$

Therefore $\phi = -\pi/6$ such that $\underline{\theta = \pi/3}$ and $\underline{\gamma = 4e^{-i\pi/6}}$.

2) We need to show that for $\epsilon > 0$ there exists a $\delta > 0$, such that [5]

$$|(2+i)z - (1+3i)| < \epsilon \qquad \text{when} \qquad |z - (1+i)| < \delta. \qquad (\epsilon\delta)$$

So we start with

$$|(2+i)z - (1+3i)| < \epsilon.$$

Using $|wz| = |w|\,|z|$ we obtain

$$|2+i| \left| z - \frac{1+3i}{2+i} \right| = \sqrt{5}\, |z - (1+i)| < \epsilon,$$

and therefore

$$|z - (1+i)| < \frac{\epsilon}{\sqrt{5}} =: \delta, \qquad (*)$$

where we introduced the constant δ by means of the last equation. Therefore working backwards, we deduce that whenever (*) holds we derive the first inequality in $(\epsilon\delta)$. Since for every ϵ the δ exists we have established the limit

$$\lim_{z \to 1+i} (2+i)z = 1 + 3i.$$

3) First we verify that $f(x,y)$ and $g(x,y)$ are harmonic functions. We compute [4]

$$\begin{aligned}
\partial_x f(x,y) &= 2e^{x^2-y^2}\left(x\cos 2xy - y\sin 2xy\right) \\
\partial_x^2 f(x,y) &= 2e^{x^2-y^2}\big(\cos 2xy + 2x^2\cos 2xy - 2y^2\cos 2xy - 4xy\sin 2xy\big) \\
\partial_y f(x,y) &= -2e^{x^2-y^2}\left(y\cos 2xy + x\sin 2xy\right) \\
\partial_y^2 f(x,y) &= -2e^{x^2-y^2}\big(\cos 2xy + 2x^2\cos 2xy - 2y^2\cos 2xy - 4xy\sin 2xy\big)
\end{aligned}$$

such that $\Delta f(x,y) = \partial_x^2 f(x,y) + \partial_y^2 f(x,y) = 0$. Next compute

$$\begin{aligned}
\partial_x g(x,y) &= \frac{y}{x^2+y^2} \\
\partial_x^2 g(x,y) &= -\frac{2xy}{(x^2+y^2)^2} \\
\partial_y g(x,y) &= -\frac{x}{x^2+y^2} \\
\partial_y^2 g(x,y) &= \frac{2xy}{(x^2+y^2)^2}
\end{aligned}$$

such that $\Delta g(x,y) = \partial_x^2 g(x,y) + \partial_y^2 g(x,y) = 0$. Hence $f(x,y)$ and $g(x,y)$ are indeed harmonic functions.

The conjugate harmonic functions $\tilde{f}(x,y)$ and $\tilde{g}(x,y)$ are obtained by solving the Cauchy-Riemann equations

$$\frac{\partial f}{\partial x} = \frac{\partial \tilde{f}}{\partial y}, \qquad \frac{\partial f}{\partial y} = -\frac{\partial \tilde{f}}{\partial x} \quad \text{and} \quad \frac{\partial g}{\partial x} = \frac{\partial \tilde{g}}{\partial y}, \qquad \frac{\partial g}{\partial y} = -\frac{\partial \tilde{g}}{\partial x}.$$

Thus [3]

$$\begin{aligned}
\partial_x f(x,y) &= 2e^{x^2-y^2}\left(x\cos 2xy - y\sin 2xy\right) = \partial_y \tilde{f}(x,y) \\
&\Rightarrow \tilde{f}(x,y) = e^{x^2-y^2}\sin 2xy + h(x)
\end{aligned}$$

and

$$\begin{aligned}
\partial_y f(x,y) &= -2e^{x^2-y^2}\left(y\cos 2xy + x\sin 2xy\right) = -\partial_x \tilde{f}(x,y) \\
&\Rightarrow \tilde{f}(x,y) = e^{x^2-y^2}\sin 2xy + \tilde{h}(y).
\end{aligned}$$

Comparing both results yields

$$\tilde{f}(x,y) = e^{x^2-y^2}\sin 2xy + c \qquad \text{with } c \in \mathbb{C}.$$

An analytic functions with $f(x,y)$ as real part is therefore

$$F(x,y) = f(x,y) + i\tilde{f}(x,y) = e^{x^2-y^2}\cos(2xy) + i\left[e^{x^2-y^2}\sin 2xy + c\right].$$

Likewise we compute [3]

$$\partial_x g(x,y) = \frac{y}{x^2+y^2} = \partial_y \tilde{g}(x,y)$$
$$\Rightarrow \tilde{g}(x,y) = \frac{1}{2}\ln(x^2+y^2) + h(x)$$

and

$$\partial_y g(x,y) = -\frac{x}{x^2+y^2} = -\partial_x \tilde{g}(x,y)$$
$$\Rightarrow \tilde{g}(x,y) = \frac{1}{2}\ln(x^2+y^2) + \tilde{h}(y).$$

Comparing both results yields

$$\tilde{g}(x,y) = \frac{1}{2}\ln(x^2+y^2) + c \quad \text{with } c \in \mathbb{C}.$$

An analytic functions with $g(x,y)$ as real part is therefore

$$G(x,y) = g(x,y) + i\tilde{g}(x,y) = \arctan\left(-\frac{y}{x}\right) + i\left[\frac{1}{2}\ln(x^2+y^2) + c\right].$$

4) We parameterize the boundary of the semi-unit disk in the upper half plane by [4]

$$e^{i\theta} \quad \text{for } 0 \leq \theta \leq \pi \qquad \text{and} \qquad r \quad \text{for } -1 \leq r \leq 1.$$

Computing

$$\left|p(e^{i\theta})\right| = \left|-i\frac{e^{i2\theta}+2ie^{i\theta}+1}{e^{i2\theta}-2ie^{i\theta}+1}\right| = \left|\frac{e^{i\theta}+2i+e^{-i\theta}}{e^{i\theta}-2i+e^{-i\theta}}\right| = \left|\frac{\cos\theta+i}{\cos\theta-i}\right| = 1$$

we see that the semi-circle in the upper half plane is mapped onto the unit circle. Evaluating a few specific points on the semi-circle in the upper half plane, including the end points

$$p(1) = 1, \quad p(e^{i\pi/6}) = \frac{1}{7}(4\sqrt{3}+i), \quad p(e^{i\pi/4}) = \frac{1}{3}(2\sqrt{2}+i), \quad p(e^{i\pi/3}) = \frac{1}{5}(4+3i),$$
$$p(i) = i, \quad p(-1) = -1,$$

we conclude that the semi-circle in the upper half plane is mapped onto the unit semi-circle in the upper half plane. Next we find the image of the line segment $-1 \leq r \leq 1$. We find [4]

$$|p(r)| = \left|\frac{r^2+2ir+1}{r^2-2ir+1}\right| = \frac{\left(r^2+1\right)^2+(2r)^2}{\left(r^2+1\right)^2+(2r)^2} = 1,$$

such that we deduce that also the line segment is mapped onto the unit circle. Evaluating a few specific points on the segment, including the end points

$$p(-1) = -1, \quad p(-1/2) = -\frac{40+9i}{41}, \quad p(0) = -i, \quad p(1/2) = \frac{40-9i}{41}, \quad p(1) = 1,$$

we conclude that the segment is mapped onto the unit semi-circle in the lower half plane.

Finally we need to establish where the semi-disk is mapped to. Since the area is connected there are only two possibilities wither to the exterior or the interior of the unit circle. In order to clarify which case occurs we just need to check where one sample point is mapped to, e.g. [2]

$$p(i/2) = \frac{i}{7}.$$

Since the image point lies within the unit circle, we conclude that the semi-unit disk in the upper half plane is mapped to the unit disk by $p(z)$.

5) We rewrite the function as [10]

$$q(z) = z\sqrt{1 - 1/z} = \sqrt{z}\sqrt{z-1}$$

and notice that it has two branch points at $z = 0$ and at $z = 1$. Parameterizing now

$$\sqrt{z} = r_1 e^{i\theta_1/2} \qquad \text{and} \qquad \sqrt{z-1} = r_2 e^{i\theta_2}, \quad \text{for}^1 \; -\pi < \theta_1, \theta_2 < \pi$$

we write $q(z)$ as

$$\begin{aligned} q(z) &= \exp\left[\ln\left(\sqrt{z}\sqrt{z-1}\right)\right] = \exp\left[\ln\left(r_1 e^{i\theta_1/2}\right) + \ln\left(r_2 e^{i\theta_2/2}\right)\right] \\ &= \exp\left[\ln(r_1) + \ln(r_2) + i\frac{\theta_1 + \theta_2}{2} + 2\pi i(n+m)\right] \qquad \text{with } n, m \in \mathbb{Z} \end{aligned}$$

The multivaluedness is taken care off by chosing $n = m = 0$. Next we have to study how the function behaves across the branch cut in order to make it analytic. For this purpose we consider different regions on the real axis:

- $z \in (-\infty, 0)$: When crossing this part of the axis both θ_1 and θ_2 are discontinuous. However, the relevant quantity, which is the difference $\theta_1 + \theta_2$ make $e^{(\theta_1+\theta_2)/2}$ a continuous function. Above the axis we have $\theta_1 = \theta_2 = \pi$, such that $\theta_1 + \theta_2 = 2\pi$ and below the axis we have $\theta_1 = \theta_2 = -\pi$ and therefore $\theta_1 + \theta_2 = -2\pi$. This means $e^{(\theta_1+\theta_2)/2}$ takes the value -1 above and below the axis, such that no cut is required on this segment.
- $z \in (0, 1)$: On this line segment θ_1 is discontinuous and θ_2 is, therefore we require a cut.
- $z \in (1, \infty)$: On this part of the axis there is no problem as θ_1 and θ_2 are all continuous when crossing the axis.

Overall we only need therefore branch cuts at the line segment $(0, 1)$ in order to make $q(z)$ single valued and analytic.