Mathematical Methods II

Coursework 2

Hand in the complete solutions to all three questions in the general office (room C123)

DEADLINE: Friday 12/12/2008 at 16:00

1) Given are the two identical functions [10 marks]

$$u(x) = v(x) = \begin{cases} 1 & \text{for } |x| < \lambda \\ 0 & \text{for } |x| > \lambda \end{cases}$$

i) Compute the convolution

$$(u \star v)(x).$$

ii) Sketch the graphs of $u(x)$, $v(x)$ and $(u \star v)(x)$.

2) Using the Laplace transformation method solve the following ordinary differential equation [20 marks]

$$\frac{d^2 u(x)}{dx^2} - \kappa^2 u(x) = e^{-\lambda x} \qquad \text{with } \kappa > 0,\ \lambda > 0,\ \kappa \neq \lambda$$

The boundary conditions are $u(0) = \alpha$ and $u'(0) = \beta$.

(You do not have to compute inverse Laplace transforms from first principles using the Bromwich integral formula, but may instead use results from the lecture.)

3) Using the Fourier transformation method solve the Schrödinger equation [20 marks]

$$i\frac{\partial \psi(x,t)}{\partial t} + \frac{\partial^2 \psi(x,t)}{\partial x^2} = 0$$

The initial condition is $\psi(x, 0) = \delta(x)$, where $\delta(x)$ denotes the Dirac delta function.

Solutions CW2 (MM08)

1) i) By definition we have

$$u * v(x) = \int_{-\infty}^{+\infty} u(s)\, v(x-s)\, ds$$

we have

$$u(s)\, v(x-s) = \begin{cases} 1 & \text{for } |x-s| < \lambda \ \wedge \ |s| < \lambda \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow u(s)\, v(x-s) = 1 \text{ for } \left.\begin{array}{l} x - s < \lambda \Rightarrow x < \lambda + s < 2s \\ -\lambda < x - s \Rightarrow s - \lambda < x \Rightarrow -2\lambda < x \end{array}\right\} \Rightarrow \underline{|x| < 2\lambda}$$

when $0 < x < 2\lambda$:

$$x - s < \lambda \Rightarrow x - \lambda < s < \lambda$$

when $-2\lambda < x < 0$:

$$-\lambda < x - s \Rightarrow -\lambda < s < x + \lambda$$

$$\Rightarrow \quad u * v(x) = \begin{cases} 0 & \text{for } x < -2\lambda \\ x + 2\lambda & \text{for } -2\lambda < x < 0 \\ 2\lambda - x & \text{for } 0 \leq x < 2\lambda \\ 0 & \text{for } x > 2\lambda \end{cases}$$

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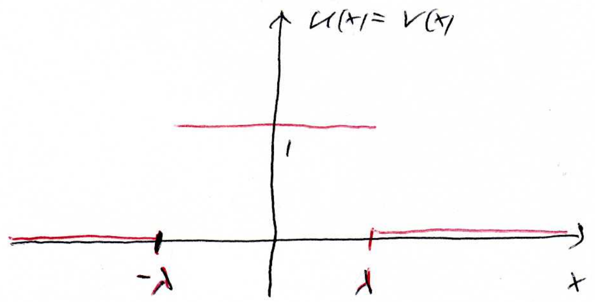
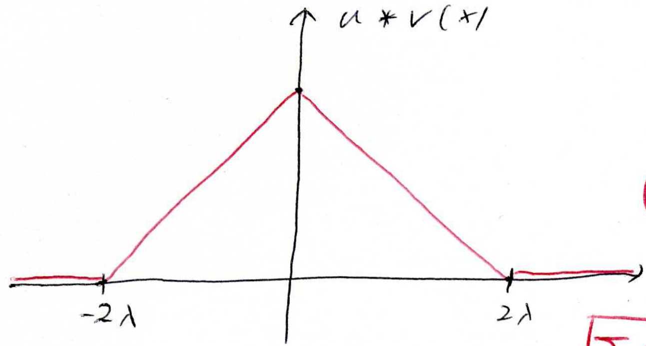
ii)





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$\boxed{\Sigma = 10}$

2)
$$u'' - k^2 u = e^{-\lambda x} \qquad k > 0,\ \lambda > 0 \quad k \neq \lambda$$

BC: $u(0) = \alpha$, $u'(0) = \beta$

$$\Rightarrow \mathcal{L} u'' - k^2 \mathcal{L} u = \mathcal{L}(e^{-\lambda x}) \qquad (*)$$

Use formula from the lecture

$$\mathcal{L} u'' = x \mathcal{L} u'(x) - u'(0) = x \left(x \mathcal{L} u(x) - u(0) \right) - u'(0)$$
$$= x^2 \mathcal{L} u(x) - x u(0) - u'(0) = x^2 \mathcal{L} u(x) - \alpha x - \beta$$

we also found in the lecture $\mathcal{L}(e^{\beta x}) = \frac{1}{x-\beta}$ for $x > \beta$

$$(*) \Rightarrow x^2 \mathcal{L}u(x) - \alpha x - \beta - k^2 \mathcal{L}u(x) = \mathcal{L}(e^{-\lambda x}) = \frac{1}{x+\lambda}$$

$$\Rightarrow \mathcal{L}u(x)\,(x^2 - k^2) - \alpha x - \beta = \frac{1}{x+\lambda}$$

$$\Rightarrow \mathcal{L}u(x) = \frac{1}{(x+\lambda)(x^2-k^2)} + \frac{\alpha x}{(x^2-k^2)} + \frac{\beta}{x^2-k^2}$$

from the lecture

$$\mathcal{L}(\sin\lambda x) = \frac{\lambda}{x^2+\lambda^2} \quad \overset{\lambda \to i\lambda}{\Longrightarrow} \quad \mathcal{L}(\sinh \lambda x) = \frac{\lambda}{x^2-\lambda^2}$$

$$\mathcal{L}(\cos\lambda x) = \frac{x}{x^2+\lambda^2} \quad \Rightarrow \quad \mathcal{L}(\cosh x) = \frac{x}{x^2-\lambda^2}$$

$$\Rightarrow \mathcal{L}u(x) = \frac{1}{k}\underbrace{\mathcal{L}(e^{-\lambda x})\,\mathcal{L}(\sinh\lambda x)}_{\mathcal{L}(e^{-\lambda *} * \sinh \lambda x) \text{ or}} + \alpha\mathcal{L}(\cosh kx) + \frac{\beta}{k}\mathcal{L}(\sinh kx)$$

$$\frac{1}{(x+\lambda)(x^2-k^2)} = \frac{1}{(k^2-\lambda^2)(x+\lambda)} + \frac{1}{2k(k+\lambda)}\frac{1}{x-k} + \frac{1}{2k(k-\lambda)}\frac{1}{x+k}$$

$$= \frac{1}{k^2-\lambda^2}\mathcal{L}(e^{-\lambda x}) + \frac{1}{2k(k+\lambda)}\mathcal{L}(e^{-kx}) + \frac{1}{2k(k-\lambda)}\mathcal{L}(e^{kx})$$

$$\Rightarrow \mathcal{L}u(x) = \frac{1}{k^2-\lambda^2}\mathcal{L}(e^{-\lambda x}) + \frac{1}{2k(k+\lambda)}\mathcal{L}(e^{-kx}) + \frac{1}{2k(k-\lambda)}\mathcal{L}(e^{kx}) + \mathcal{L}(\cosh kx)$$
$$+ \mathcal{L}(\sinh kx)$$

$$\Rightarrow u(x) = \alpha\cosh kx + \frac{\beta}{k}\sinh kx + \frac{e^{-\lambda x}}{k^2-\lambda^2} + \underbrace{\frac{1}{2k(k+\lambda)}e^{-kx} + \frac{1}{2k(k-\lambda)}e^{k}}_{\frac{\cosh kx + \frac{\lambda}{k}\sinh(kx)}{k^2-\lambda^2}}$$

$\boxed{\Sigma = 20}$

3/ Solve $i\psi_t(x,t) + \psi_{xx}(x,t) = 0$ (1) with $\psi(x,0) = \delta(x)$

Fourier transform: $\hat{\psi}(x,t) = \int_{-\infty}^{+\infty} \psi(s,t)\, e^{-isx}\, ds$

$$\Rightarrow i\partial_t \hat{\psi}(x,t) = i\int_{-\infty}^{+\infty} \partial_t \psi(s,t)\, e^{-isx}\, ds \overset{(1)}{=} -\int_{-\infty}^{+\infty} \psi_{ss}(s,t)\, e^{-isx}\, ds$$

$$= -\psi_s(s,t)\, e^{-isx}\Big|_{-\infty}^{+\infty} - ix\int_{-\infty}^{+\infty} \psi_s\, e^{-isx}\, ds \quad \text{(int. by parts)}$$

$$= \underbrace{-\psi_s(s,t)\, e^{-isx}\Big|_{-\infty}^{+\infty}}_{0} \underbrace{- ix\,\psi(s,t)\, e^{-isx}\Big|_{-\infty}^{+\infty}}_{0} + x^2\int_{-\infty}^{+\infty} \psi(s,t)\, e^{-isx}\, ds$$

with $\lim_{x\to\pm\infty} \psi(x,t) = \lim_{x\to\pm\infty} \psi_x(x,t) = 0$ physical boundary con

$$\Rightarrow i\partial_t \hat{\psi}(x,t) = x^2\, \hat{\psi}(x,t) \quad \text{with initial condition } \hat{\psi}(x,0) = \hat{\delta}(x)$$

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$$\Rightarrow \text{solution } \hat{\psi}(x,t) = e^{-itx^2}\, \hat{\delta}(x)$$

recall $\mathcal{F}(u(x)) = \sqrt{\pi}\, e^{-\frac{x^2}{4}}$ for $u(x) = e^{-x^2}$ (Ex 5 sec. 3

$\mathcal{F}(v(x)) = 2\sqrt{\pi t}\, e^{-tx^2}$ for $v(x) = e^{-\frac{x^2}{4t}}$ (sec. 3.1

$\Rightarrow \mathcal{F}(w(x)) = e^{-itx^2}$ for $w(x) = \frac{1}{2\sqrt{i\pi t}}\, e^{\frac{ix^2}{4t}}$

$$\Rightarrow \hat{\psi}(x,t) = \hat{w}(x,t)\, \hat{\delta}(x)$$

$$\Leftrightarrow \mathcal{F}_x\, \psi(x,t) = \mathcal{F}_x\, w(x,t)\, \mathcal{F}_x\, \delta(x) = \mathcal{F}_x\big(w(x,t) * \delta(x)\big)$$

$$\Rightarrow \underline{\underline{\psi(x,t)}} = w(x,t) * \delta(x) = \underline{\underline{\int_{-\infty}^{+\infty} dy\, \delta(y)\, \frac{1}{2\sqrt{i\pi t}}\, e^{\frac{i}{4t}(x-y)^2}}}$$

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$\boxed{\Sigma = 20}$