Mathematical Methods II

Coursework 2

Hand in the complete solutions to all three questions in the general office(room C123).

DEADLINE: Tuesday 8/12/2009 at 16:00

1) The Laguerre polynomials $L_n(x)$ are generated from [15 marks]

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right).$$

Compute the Fourier transform for the function

$$u(x) = 6L_3(x)e^{-x^2}.$$

2) *i)* Compute the Laplace transformation for the function [20 marks]

$$u(x) = e^{-x}\sin(\alpha x) \qquad \text{for } \alpha \in \mathbb{R}.$$

ii) Use the Laplace transformation method to solve the following ordinary differential equation

$$\frac{d^2y(x)}{dx^2} + 2\frac{dy(x)}{dx} + 5y(x) = 3e^{-x}\sin(x).$$

The boundary conditions are $y(0) = 0$ and $dy/dx(0) = 3$.

(You do not have to compute inverse Laplace transforms from first principles using the Bromwich integral formula, but may instead use the result from *i*).)

iii) By substitution verify that your solution is correct.

3) Use the Laplace transformation method to solve the integral equation [15 marks]

$$y(x) = x^3 + \int_0^x \sin(x-t)y(t)dt.$$

CW II Mathematical Methods II (Solutions)

1) Compute the Laguerre polynomial $L_3(x)$:

$$L_3(x) = \frac{e^x}{3!}\frac{d^3}{dx^3}\left(e^{-x}x^3\right) = \frac{e^x}{6}\frac{d^2}{dx^2}\left(-e^{-x}x^3 + e^{-x}3x^2\right)$$

$$= \frac{e^x}{6}\frac{d}{dx}\left(-e^{-x}(3x^2 - x^3) + e^{-x}(6x - 3x^2)\right)$$

$$= \frac{e^x}{6}\frac{d}{dx}\left[e^{-x}(x^3 - 6x^2 + 6x)\right] = \frac{e^x}{6}\left(-e^{-x}(x^3-6x^2+6x) + e^{-x}(3x^2-12x+6)\right)$$

$$L_3(x) = \frac{1}{6}\left(6 - 18x + 9x^2 - x^3\right) \quad \Rightarrow \quad u(x) = \left(6 - 18x + 9x^2 - x^3\right)e^{-x^2}$$

$$v(x) = e^{-x^2}, \quad v'(x) = -2x e^{-x^2}, \quad v''(x) = (4x^2-2)e^{-x^2}, \quad v'''(x) = (12x - 8x^3)e^{-x^2}$$

assume: $u(x) = A\,v(x) + B\,v'(x) + C\,v''(x) + D\,v'''(x)$

$$= \left[(A - 2C) + (-2B + 12D)x + 4Cx^2 - 8Dx^3\right]e^{-x^2}$$

$$\Rightarrow \quad A - 2C = 6 \qquad 12D - 2B = -18 \qquad 4C = 9 \qquad -8D = -1$$

$$\Rightarrow \quad D = \frac{1}{8} \qquad C = \frac{9}{4} \qquad B = \frac{39}{4} \qquad A = \frac{21}{2}$$

we know $\mathcal{F}u'(x) = ix\,\mathcal{F}u(x)$ $\qquad \mathcal{F}v(x) = \sqrt{\pi}\,e^{-\frac{x^2}{4}}$ for $v(x) = e^{-x^2}$

$$\Rightarrow \mathcal{F}u(x) = A\,\mathcal{F}v(x) + B\,ix\,\mathcal{F}v(x) + C\,(ix)^2\,\mathcal{F}v(x) + D\,(ix)^3\,\mathcal{F}v(x)$$

$$\mathcal{F}u(x) = \left(\frac{21}{2} + ix\frac{39}{4} - x^2\frac{9}{4} - \frac{i}{8}x^3\right)\sqrt{\pi}\,e^{-\frac{x^2}{4}}$$ 15

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2) i) Compute $\mathcal{L}u(x)$ with $u(x) = e^{-x}\sin(\alpha x)$

$$\mathcal{L}u(x) = \int_0^\infty e^{-t}\sin(\alpha t)\,e^{-tx}\,dx = \int_0^\infty \underset{u'}{\sin(\alpha t)}\,\underset{v}{e^{-(1+x)t}}\,dt$$

$u = -\frac{1}{\alpha}\cos\alpha t$

$v' = -(1+x)e^{-(1+x)t}$

$$= e^{-(1+x)t}\left(-\frac{1}{\alpha}\right)\cos\alpha t\Big|_0^\infty - \frac{1+x}{\alpha}\int_0^\infty \underset{u'}{\cos(\alpha t)}\,\underset{v}{e^{-(1+x)t}}\,dt$$

$u = \frac{1}{\alpha}\sin(\alpha t)$

$$= \frac{1}{\alpha} + \underbrace{\left(\frac{1+x}{\alpha}\right)\frac{1}{\alpha}\sin(\alpha t)\,e^{-(1+x)t}\Big|_0^\infty}_{0} - \frac{(1+x)^2}{\alpha^2}\underbrace{\int_0^\infty \sin\alpha t\,e^{-(1+x)t}\,dt}_{\mathcal{L}u(x)}$$

$$= \frac{1}{\alpha} - \frac{(1+x)^2}{\alpha^2}\mathcal{L}u(x) \quad \Rightarrow \quad \mathcal{L}u(x) = \frac{\alpha}{\alpha^2 + (1+x)^2}$$ 8

ii)

$$y'' + 2y' + 5y = 3e^{-x}\sin x$$

$$\mathcal{L}y'(x) = x\,\mathcal{L}y(x) - y(0)$$

$$\mathcal{L}y''(x) = x\,\mathcal{L}y'(x) - y'(0) = x\left(x\,\mathcal{L}y(x) - y(0)\right) - y'(0) = x^2\mathcal{L}y(x) - x\,y(0) - y'(0)$$

$\Rightarrow \quad \mathcal{L}y'' + 2\mathcal{L}y' + 5\mathcal{L}y = 3\mathcal{L}(e^{-x}\sin x)$

$\Rightarrow \quad x^2 \mathcal{L}y(x) - 3 + 2x\mathcal{L}y(x) + 5\mathcal{L}y(x) = 3\mathcal{L}(e^{-x}\sin x)$

$\Rightarrow \quad \mathcal{L}y(x)(x^2+2x+5) = \frac{3}{1^2+(1+x)^2} + 3$

$\Rightarrow \quad \mathcal{L}y(x) = \left(3 + \frac{3}{1+(1+x)^2}\right)\frac{1}{x^2+2x+5}$

$= \frac{3}{1+(1+x)^2} + \frac{3}{(x^2+2x+2)(x^2+2x+5)}$

$= \frac{3}{1+(1+x)^2} + \frac{1}{x^2+2x+2} - \frac{1}{x^2+2x+5}$

$= \frac{2}{1+(1+x)^2} + \frac{1}{1+(1+x)^2} = \mathcal{L}(\sin 2x\, e^{-x}) + \mathcal{L}(\sin x\, e^{-x})$

$\Rightarrow \quad y(x) = e^{-x}(\sin x + \sin 2x)$ 7

iii)

$y'(x) = e^{-x}(\cos x + 2\cos 2x) - e^{-x}(\sin x + \sin 2x)$

$= e^{-x}(-\sin x + \cos x + 2\cos 2x - \sin 2x)$

$y''(x) = -e^{-x}(-\sin x + \cos x + 2\cos 2x - \sin 2x)$

$+ e^{-x}(-\cos x - \sin x - 4\sin 2x - 2\cos 2x)$

$= e^{-x}(2\cos x - 3\sin 2x - 4\cos 2x)$

$\Rightarrow e^{-x}(-2\cos x - 3\sin 2x - 4\cos 2x) + 2e^{-x}(-\sin x + \cos x + 2\cos 2x - \sin 2x)$

$+ 5e^{-x}(\sin x + \sin 2x) = 3e^{-x}\sin x$ ✓

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boundary conditions: $y(0) = 0 \qquad y'(0) = 3$ ✓

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3)

$y(x) = x^3 + \int_0^x \sin(x-t)\, y(t)\, dt$

$= x^3 + \int_0^\infty H(x-t)\sin(x-t)\, y(t)\, dt$

$= x^3 + v * y(x) \qquad v(x) = H(x)\sin(x)$

$\Rightarrow \quad \mathcal{L}y(x) = \mathcal{L}x^3 + \mathcal{L}(v*y)(x)$

$\Rightarrow \quad \mathcal{L}y(x) = \mathcal{L}x^3 + \mathcal{L}v(x)\,\mathcal{L}y(x)$

$$\Rightarrow \quad \mathcal{L}y(x)\,(1-\mathcal{L}v(x)) = \mathcal{L}x^3$$

$$\Rightarrow \quad \mathcal{L}y(x) = \frac{\mathcal{L}x^3}{1-\mathcal{L}v(x)}$$

with

$$\left.\begin{aligned} \mathcal{L}x^3 &= \int_0^\infty t^3 e^{-tx}\,dt = \frac{6}{x^4} \\ \mathcal{L}v(x) &= \int_0^\infty H(t)\sin t\, e^{-tx}\,dt = \frac{1}{1+x^2} \end{aligned}\right\} \Rightarrow \mathcal{L}y(x) = \frac{6/x^4}{1-\frac{1}{1-x^2}} = \frac{\frac{6(1+x^2)}{x^4}}{1+x^2-1}$$

$$\Rightarrow \quad \mathcal{L}y(x) = \frac{6(1+x^2)}{x^6} = \frac{6}{x^6} + \frac{6}{x^4} = \mathcal{L}x^3 + \frac{1}{20}\mathcal{L}x^5$$

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$$\Rightarrow \quad y(x) = x^3 + \frac{1}{20}x^5$$

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$$\boxed{\Sigma = 50}$$