Mathematical Methods II

Coursework 2

Hand in the complete solutions to all two questions in the general office by Thursday 06/12/2007 at 16:00

1) (25 marks)

By computing the Bromwich integral

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} v(t)e^{tx}dt \qquad \text{for } \gamma > \alpha,$$

where α is the exponential growth of $\mathcal{L}^{-1}v(x)$, find the inverse Laplace transform for the function

$$v(x) = \frac{x\omega}{(x^2+\omega^2)^2} \qquad \text{for } \omega \in \mathbb{R}.$$

2) (25 marks)

i) Prove the following formula

$$\mathcal{L}u^{(n)}(x) = x^n\mathcal{L}u(x) - \sum_{k=0}^{n-1} x^{n-k-1}u^{(k)}(0),$$

which relates the Laplace transform of the n-th derivative of a function $\mathcal{L}u^{(n)}(x)$ to the Laplace transform of the function itself $\mathcal{L}u(x)$.

ii) Using a similar formula as the one in i) for Fourier transforms, compute the Fourier transform of

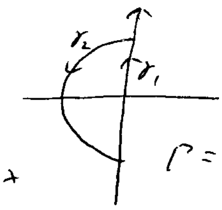
$$u(x) = (7 + 20x - 12x^2 - 16x^3)e^{-x^2}.$$

You may use the integral $\mathcal{F}v(x) = \sqrt{\pi}e^{-x^2/4}$ for $v(x) = e^{-x^2}$.

Solutions CW 2

Compute $\mathcal{L}^{-1} v(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dt \, \frac{t\omega}{(t^2+\omega^2)^2} \, e^{tx}$

Parameterise $z = \varepsilon + r e^{i\theta}$



$$\Rightarrow \frac{1}{2\pi i} \oint_{\Gamma} \frac{z\omega e^{zx}}{(z^2+\omega^2)^2} \, dz \underset{\substack{\uparrow \\ \text{Residue theorem}}}{=} \frac{2\pi i}{2\pi i} \operatorname*{Res}_{z_0 = \pm i\omega} \frac{z_0 \omega e^{z_0 x}}{(z_0 - i\omega)^2 (z_0 + i\omega)^2}$$

with $\operatorname*{Res}_{z_0=\pm i\omega} \frac{z_0 \omega e^{z_0 x}}{(z_0 - i\omega)^2 (z_0 + i\omega)^2} = \lim_{z \to \pm i\omega} \frac{d}{dz} \Big((z \mp i\omega)^2 \frac{z\omega e^{z}}{(z-i\omega)^2 (}$

$$= \lim_{z \to \pm i\omega} \frac{d}{dz} \left(\frac{z\omega e^{zx}}{(z \pm i\omega)^2} \right)$$

$$= \lim_{z \to \pm i\omega} \left(- \frac{2 e^{zx} z\omega}{(z \pm i\omega)^3} + \frac{e^{zx}}{(z \pm i\omega)^2} + \frac{e^{zx} \, x z\omega}{(z \pm i\omega)^2} \right)$$

$$= \mp \frac{i}{4} \, x \, e^{\pm i x\omega}$$

$$\Rightarrow \frac{1}{2\pi i} \oint \frac{z\omega e^{zx}}{(z^2+\omega^2)^2} \, dz = -\frac{i}{4} x \left(e^{ix\omega} - e^{-ix\omega} \right) = \frac{1}{2} x \sin \omega$$

We need to show that $\oint_{\gamma_2} \to 0$ for $r \to \infty$

$$\left| \int_{\gamma_2} \frac{z\omega e^{zx}}{(z^2+\omega^2)^2} \, dz \right| = \omega e^{\varepsilon x} \left| \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta \, \frac{r e^{i\theta} (\varepsilon + r e^{i\theta}) e^{r e^{i\theta}}}{\left[(\varepsilon + r e^{i\theta})^2 + \omega^2 \right]^2} \right.$$

$$< \omega e^{\varepsilon x} \frac{(\varepsilon + r) r}{(r^2 + \omega^2)^2} \to 0 \quad \text{for } r \to$$

with $|r e^{i\theta}| = r$; $|\varepsilon + r e^{i\theta}| \le \varepsilon + |r e^{i\theta}| = \varepsilon + r$;

$|e^{x r e^{i\theta}}| = e^{x r \cos\theta} \le 1$ for $\frac{\pi}{2} \le \theta \le \frac{3}{2}\pi$

$\left[(\varepsilon + r e^{i\theta})^2 + \omega^2 \right]^2 > (r^2 + \omega^2)^2$

$$\Rightarrow \mathcal{L}^{-1} v(x) = \frac{1}{2} x \sin \omega x$$

$n=1$

$$\mathcal{L}u'(x) = x\mathcal{L}u(x) - u(0)$$

$$= \int_0^\infty u'(t)e^{-tx}\,dt$$

$$= u(t)e^{-tx}\Big|_0^\infty + x\int_0^\infty u(t)e^{-tx}\,dx$$

$$= x\mathcal{L}u(x) - u(0)$$

$$\mathcal{L}u^{(n)}(x) = \int_0^\infty u^{(n)}(x)e^{-tx}\,dx$$

$$= u^{(n-1)}(x)e^{-tx}\Big|_0^\infty + x\int_0^\infty u^{n-1}(t)e^{-tx}\,dt$$

$$= -u^{(n-1)}(0) + x\mathcal{L}u^{(n-1)}(x)$$

$$\mathcal{L}u^{(n-1)}(x) = -u^{(n-2)}(0) + x\mathcal{L}^{(n-2)}(x)$$

$$\mathcal{L}u^{(n-2)}(x) = -u^{(n-3)}(0) + x\mathcal{L}^{(n-3)}(x)$$

$$\vdots$$

$$\mathcal{L}u^{(n)}(x) = x^n\mathcal{L}u(x) - \sum_{k=0}^{n-1} x^{n-k-1}u^{(k)}(0) \qquad (1$$

We have:

$$\mathcal{F}u'(x) = ix\mathcal{F}u(x)$$

$$\mathcal{F}u^{(2)}(x) = ix\mathcal{F}u'(x) = (ix)^2\mathcal{F}u(x) = -x^2\mathcal{F}u(x$$

$$\mathcal{F}u^{(3)}(x) = ix\mathcal{F}u^{(2)}(x) = -ix^3\mathcal{F}u(x)$$

$$\vdots$$

$$\mathcal{F}u^{(n)}(x) = (ix)^n\mathcal{F}u(x)$$

$v(x) = e^{-x^2}$

$v'(x) = -2\, e^{-x^2} x$

$v''(x) = (-2 + 4x^2)\, e^{-x^2}$

$v'''(x) = (12x - 8x^3)\, e^{-x^3}$

$u(x) = a\, v(x) + b\, v'(x) + c\, v''(x) + d\, v'''(x)$

$= (7 + 20x - 12x^2 - 16x^3)\, e^{-x^2}$

$= \left[(a - 2c) + (12d - 2b)x + 4cx^2 - 8dx^3\right] e^{-x^2}$

$\Rightarrow \quad d = 2 \; ; \quad c = -3 \; ; \quad a = 1 \; , \quad b = 2$

$\Rightarrow \quad \mathcal{F} u(x) = \mathcal{F} v(x) + 2\, \mathcal{F} v'(x) - 3\, \mathcal{F} v''(x) + 2\, \mathcal{F} v'''(x)$

$= \mathcal{F} v(x) + 2\, ix\, \mathcal{F} v(x) - 3\, (ix)^2\, \mathcal{F} v(x) + 2\, (ix)^3\, \mathcal{F} v(x)$

$= (1 + 2ix + 3x^2 - i2x^3)\, \mathcal{F} v(x)$

$= (1 + 2ix + 3x^2 - 2ix^3)\, \sqrt{\pi}\, e^{-\frac{x^2}{4}}$

(15)

$\boxed{\Sigma = 25}$