Mathematical Methods II

Coursework 2

Hand in the complete solutions to all three questions to the SEMS general office (C109).

DEADLINE: Friday 08/04/2011 at 14:00

1) (10 marks) Use the relation between the Fourier transform and the Fourier transform of its derivative $\mathcal{F}u'(x) = ix\mathcal{F}u(x)$ together with $\int_{-\infty}^{\infty} e^{-t^2} e^{-itx} dt = \sqrt{\pi} e^{-x^2/4}$ to compute the Fourier transform $\mathcal{F}u(x)$ of the function

$$u(x) = 8P_4(x)e^{-x^2},$$

where $P_4(x)$ is a Legendre polynomial of order four. The Legendre polynomial of order n can be generated from

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n .$$

2) (25 marks) Compute the inverse Laplace transform $\mathcal{L}^{-1}v(x)$ for the function

$$v(x) = \frac{7}{x^2 - 49}$$

in two alternative ways.

i) By evaluating the Bromwich integral

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} v(t) e^{tx} dt \qquad \text{for } \gamma > \alpha,$$

where α is the exponential growth of $\mathcal{L}^{-1}v(x)$.

ii) By using some known Laplace transforms from the lecture.

3) (15 marks) Show, using Laplace transforms, that the solution to equation

$$my''(x) + \lambda y'(x) + \kappa y(x) = F(x)$$

subject to the initial conditions $y(0) = y'(0) = 0$ is of the form

$$y(x) = \frac{1}{m\omega} \int_0^{\infty} ds\ F(x - s) e^{-\mu s} \sin \omega s.$$

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Solutions and marking scheme

DEADLINE: Friday 08/04/2011 at 14:00

1) First we compute the Legendre polynomial of order four $P_4(x)$ [10 marks]

$$\begin{aligned} P_4(x) &= \frac{1}{2^4 4!}\frac{d^4}{dx^4}\left(x^2-1\right)^4 \\ &= \frac{1}{48}\frac{d^3}{dx^3}\left[x\left(x^2-1\right)^3\right] \\ &= \frac{1}{48}\frac{d^2}{dx^2}\left[\left(x^2-1\right)^2\left(7x^2-1\right)\right] \\ &= \frac{1}{8}\frac{d}{dx}\left(7x^5-10x^3+3x\right) \\ &= \frac{1}{8}\left(35x^4-30x^2+3\right), \end{aligned}$$

such that

$$u(x) = \left(35x^4-30x^2+3\right)e^{-x^2}.$$

Next we define $v(x) = e^{-x^2}$. Then $v'(x) = -2xe^{-x^2}$, $v''(x) = (4x^2-2)e^{-x^2}$, $v'''(x) = (12x-8x^3)e^{-x^2}$ and $v^{iv}(x) = 4(3-12x^2+4x^4)e^{-x^2}$. Expanding $u(x)$ therefore as

$$u(x) = Av(x) + Bv'(x) + Cv''(x) + Dv'''(x) + Ev^{iv}(x)$$

and comparing coefficients we find

$$A-3-2C+12E=0,\ 12D-2B=0,\ 30+4C-48E=0,\ -8D=0,\ 16E-35=0$$

Solving these equations gives

$$u(x) = \frac{57}{4}v(x) + \frac{75}{4}v''(x) + \frac{35}{16}v^{iv}(x).$$

Therefore

$$\begin{aligned} \mathcal{F}u(x) &= \frac{57}{4}\mathcal{F}v(x) + \frac{75}{4}\mathcal{F}v''(x) + \frac{35}{16}\mathcal{F}v^{iv}(x) \\ &= \frac{57}{4}\mathcal{F}v(x) - \frac{75}{4}x^2\mathcal{F}v(x) + \frac{35}{16}x^4\mathcal{F}v(x) \\ &= \sqrt{\pi}\left[\frac{57}{4} - \frac{75}{4}x^2 + \frac{35}{16}x^4\right]e^{-x^2/4} \\ &= \frac{\sqrt{\pi}}{16}\left[228-300x^2+35x^4\right]e^{-x^2/4} \end{aligned}$$

We used here repeatedly the relation between the Fourier transform and the Fourier transform of its derivative $\mathcal{F}u'(x) = ix\mathcal{F}u(x)$ together with $\int_{-\infty}^{\infty} e^{-t^2} e^{-itx} dt = \sqrt{\pi} e^{-x^2/4}$.

2) i) Compute [25 marks]

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{7}{t^2 - 49} e^{tx} dt.$$

Parameterize $z = \varepsilon + re^{i\theta}$ and compute [20]

$$\frac{1}{2\pi i} \oint_\Gamma \frac{7}{z^2 - 49} e^{zx} dz = \frac{2\pi i}{2\pi i} \operatorname*{Res}_{z_0 = \pm\omega} \frac{7}{(z-7)(z+7)} e^{zx} = \frac{1}{2} e^{7x} - \frac{1}{2} e^{-7x} = \sinh 7x$$

In order to show that

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{t}{t^2 - 49} e^{tx} dt = \oint_\Gamma \frac{z}{z^2 - 49} e^{zx} dz$$

we have to guarantee that the integral over, say Γ, parameterized by $re^{i\theta}$ for θ from $\pi/2$ to $3\pi/2$ vanishes as $r \to \infty$. Compute

$$\left| \oint_\gamma \frac{7}{z^2 - 49} e^{zx} dz \right| = \left| 7e^{\varepsilon x} \int_{\pi/2}^{3\pi/2} \frac{re^{i\theta}}{(\varepsilon + re^{i\theta})^2 - 49} e^{re^{i\theta} x} d\theta \right|$$

With

$$\left| re^{i\theta} \right| = r$$

$$\left| e^{re^{i\theta} x} \right| = e^{rx\cos\theta} \leq 1 \qquad \text{for } \frac{\pi}{2} \leq \theta \leq \frac{3}{2}\pi$$

$$\left| (\varepsilon + re^{i\theta})^2 - 49 \right| > r^2 - 49$$

follows

$$\left| e^{\varepsilon x} \int_{\pi/2}^{3\pi/2} \frac{re^{i\theta}}{(\varepsilon + re^{i\theta})^2 - 49} e^{re^{i\theta} x} d\theta \right| < e^{\varepsilon x} \frac{r}{r^2 - 49} \to 0 \quad \text{for } r \to \infty.$$

$$\Rightarrow \mathcal{L}^{-1}v(x) = \sinh 7x$$

ii) We can write [5]

$$v(x) = \frac{7}{x^2 - 49} = \frac{7}{(x-7)(x+7)} = \frac{1}{2} \left(\frac{1}{x-7} - \frac{1}{x+7} \right)$$

From the lecture we know that for $u(x) = \exp(\alpha x)$

$$\mathcal{L}u(x) = \frac{1}{x - \alpha} \qquad \Leftrightarrow \qquad \mathcal{L}^{-1} \frac{1}{x - \alpha} = \exp(\alpha x).$$

Therefore

$$\mathcal{L}^{-1}v(x) = \frac{1}{2} \left(\mathcal{L}^{-1} \frac{1}{x-7} - \mathcal{L}^{-1} \frac{1}{x+7} \right) = \frac{1}{2} \left(e^{7x} - e^{-7x} \right) = \sinh 7x.$$

3) Acting with the Laplace operator $\mathcal{L}$ on [15 marks]

$$m\ddot{x}(t) + \lambda\dot{x}(t) + \kappa x(t) = F(t)$$

gives

$$m\mathcal{L}\ddot{x}(t) + \lambda\mathcal{L}\dot{x}(t) + \kappa\mathcal{L}x(t) = \mathcal{L}F(t). \qquad (1)$$

Applying $\mathcal{L}u^{(n)}(x) = x^n\mathcal{L}u(x) - \sum_{k=0}^{n-1} x^{n-k-1}u^{(k)}(0)$ for $n = 1, n = 2$ we find

$$\mathcal{L}\ddot{x}(t) = t^2\mathcal{L}x(t) - tx(0) - \dot{x}(0) \qquad \Rightarrow \qquad \mathcal{L}\ddot{x}(t) = t^2\mathcal{L}x(t)$$
$$\mathcal{L}\dot{x}(t) = t\mathcal{L}x(t) - x(0) \qquad \Rightarrow \qquad \mathcal{L}\dot{x}(t) = t\mathcal{L}x(t),$$

We used the initial conditions $x(0) = \dot{x}(0) = 0$. Therefore we can rewrite (1) as

$$mt^2\mathcal{L}x(t) + \lambda t\mathcal{L}x(t) + \kappa\mathcal{L}x(t) = \mathcal{L}F(t),$$

which we can solve for $\mathcal{L}x(t)$

$$\mathcal{L}x(t) = \frac{\mathcal{L}F(t)}{mt^2 + \lambda t + \kappa} = \frac{\mathcal{L}F(t)}{m(t^2 + \lambda/mt + \kappa/m)} = \frac{\mathcal{L}F(t)}{m}\frac{1}{(t+\mu)^2 + \omega^2}. \qquad (2)$$

Here we completed the square and abbreviated $\omega^2 = \kappa/m - \mu^2, \mu = \lambda/2m$. Using the hint

$$\mathcal{L}u(x) = \frac{\lambda}{(x-\mu)^2 + \lambda^2} \qquad \text{for} \qquad u(x) = e^{\mu x}\sin\lambda x.$$

Translating this to our notation here gives

$$\mathcal{L}v(t) = \frac{\omega}{(t+\mu)^2 + \omega^2} \qquad \text{for} \qquad v(t) = e^{-\mu t}\sin\omega t. \qquad (3)$$

This means we can rewrite (2) as

$$\mathcal{L}x(t) = \frac{1}{m\omega}\mathcal{L}F(t)\mathcal{L}v(t) = \frac{1}{m\omega}\mathcal{L}(F{*}v)(t), \qquad (4)$$

where we used $\mathcal{L}(u \star v)(x) = (\mathcal{L}u)(x)(\mathcal{L}v)(x)$ in the last equality. Acting now with $\mathcal{L}^{-1}$ on (4) yields the final asnwer for $x(t)$ in form of an integral representation

$$x(t) = \frac{1}{m\omega}F{*}v(t) = \frac{1}{m\omega}\int_0^\infty ds\ F(t-s)e^{-\mu s}\sin\omega s. \qquad (5)$$