Mathematical Methods II[1]

Exercises 1

(This should be revision material.)

1) Represent the following complex numbers in the form $z = x + iy$ with $x, y \in \mathbb{R}$. Determine also their absolute values, arguments and thus write them in polar form.

$i)\quad z = (-i)^5 \qquad ii)\quad z = (1+i)^4 \qquad iii)\quad z = (\sqrt{5} - i) - i(2 - i\sqrt{5})$

$iv)\quad z = \frac{10}{(1-i)(2-i)(3-i)} \qquad v)\quad z = \frac{(1+i)^5}{(1-i)^5} \qquad vi)\quad z = \frac{1}{(3-i)^2}$

2) Prove the triangle inequalities

$$|z| - |w| \leq |z + w| \leq |z| + |w| \, .$$

3) Find all fixed points of the functions

$$f_1(z) = \frac{2z - 2}{z}, \quad f_2(z) = \frac{1}{z^2}, \qquad \text{for } z \neq 0.$$

4) Provide a reasoning for the derivatives for the following functions

$$\begin{array}{lll} f_1(z) = 2xy + xy^2 i & f_2(z) = z - \bar{z} & f_3(z) = 1 + \bar{z} \\ f_4(z) = 1 + z & f_5(z) = \frac{1}{z} & f_6(z) = f_4'(z) \end{array}$$

to exist. In case they do exist, compute $f_i'(z)$ for $i = 1, 2, \ldots, 6$.

5) Show that the functions

$$u_1(x, y) = 2x - 2xy \qquad \text{and} \qquad u_2(x, y) = x^3 - \lambda x y^2, \ \lambda \in \mathbb{R}$$

are harmonic functions. Construct their conjugate harmonic functions and therefore an analytic function.

[1] The course material can be found on: http://www.staff.city.ac.uk/~fring/MathMeth/

Solutions to exercises 1

1)

$$\begin{array}{llllll} i) & z = e^{-i\pi/2} & ii) & z = 4e^{i\pi} & iii) & z = -3i = 3e^{-i\pi/2} \\ iv) & z = i = e^{i\pi/2} & v) & z = i & vi) & z = \frac{2}{25} + \frac{3}{50} i = \frac{1}{10} e^{i \arctan 3/4} \end{array}$$

2) Start with

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w}) = z\bar{z} + w\bar{z} + z\bar{w} + w\bar{w} = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}).$$

Therefore

$$|z+w|^2 - (|z|+|w|)^2 = -2\,|z|\,|w| + 2\,\mathrm{Re}(z\bar{w})$$

but $\mathrm{Re}(z\bar{w}) \leq |z\bar{w}| = |z|\,|w|$, such that

$$|z+w|^2 - (|z|+|w|)^2 \leq 0.$$

3) The fixed points are

$$f_1 : z_f^{(1/2)} = 1 \pm i, \quad f_2 : z_f^{(1/2)} = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

4) $f_1'(z)$, $f_2'(z)$ and $f_3'(z)$ do not exist because the Cauchy-Riemann condition do not hold in general

$$\begin{array}{ll} \frac{\partial u_1}{\partial x} = 2y \neq \frac{\partial v_1}{\partial y} = 2yx & \frac{\partial u_1}{\partial y} = 2x \neq -\frac{\partial v_1}{\partial x} = -y^2 \\ \frac{\partial u_2}{\partial x} = 0 \neq \frac{\partial v_2}{\partial y} = 2 & \frac{\partial u_2}{\partial y} = 0 = -\frac{\partial v_2}{\partial x} \\ \frac{\partial u_3}{\partial x} = 1 \neq \frac{\partial v_3}{\partial y} = -1 & \frac{\partial u_3}{\partial y} = 0 = -\frac{\partial v_3}{\partial x} \end{array}$$

$f_4'(z)$ and $f_5'(z)$ do exist, because the Cauchy-Riemann conditions hold and the partial derivatives are continuous

$$\begin{array}{ll} \frac{\partial u_4}{\partial x} = 1 = \frac{\partial v_4}{\partial y} & \frac{\partial u_4}{\partial y} = 0 = -\frac{\partial v_4}{\partial x} \Rightarrow f_4'(z) = 1 \\ \frac{\partial u_5}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v_5}{\partial y} & \frac{\partial u_5}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = -\frac{\partial v_5}{\partial x} \\ \Rightarrow f_5'(z) = \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} = -\frac{1}{z^2} \quad \text{for } z \neq 0 & \end{array}$$

$f_6'(z)$ does exist, because $f_4(z)$ is an analutic function and therfore also $f_4'(z)$ and $f_4''(z)$. $f_6'(z) = 0$.

5) Clealy $\Delta u_1(x,y) = \partial_x^2 u_1(x,y) + \partial_y^2 u_1(x,y) = 0$. Then

$$\left.\begin{array}{ll} \partial_x u_1(x,y) = 2 - 2y = \partial_y v_1(x,y) & \Rightarrow v_1(x,y) = 2y - y^2 + f(x) \\ \partial_y u_1(x,y) = -2x = -\partial_x v_1(x,y) & \Rightarrow v_1(x,y) = x^2 + \tilde{f}(y) \end{array}\right\} \Rightarrow v_1(x,y) = x^2 - y^2 + 2y + c$$

$$\left.\begin{array}{l} \partial_x^2 u_2(x,y) = \partial_x(3x^2 - \lambda y^2) = 6x \\ \partial_y^2 u_2(x,y) = \partial_y(-2\lambda xy) = -2\lambda x \end{array}\right\} \Rightarrow \Delta u_2(x,y) = 0 \quad \text{for } \lambda = 3$$

$$\left.\begin{array}{ll} \partial_x u_2(x,y) = 3x^2 - 3y^2 = \partial_y v_2(x,y) & \Rightarrow v_2(x,y) = 3x^2 y - y^3 + f(x) \\ \partial_y u_2(x,y) = -6xy = -\partial_x v_2(x,y) & \Rightarrow v_2(x,y) = 3x^2 y + \tilde{f}(y) \end{array}\right\} \Rightarrow v_2(x,y) = 3x^2 y - y^3 + c$$