Mathematical Methods II

Exercises 2

1) Consider the map $w = f(z) = z^2$. Determine to which kind of geometrical configuration straight horizontal and vertical lines in the w-plane correspond to in the z-plane? Sketch the corresponding figures.

2) Find the linear fractional transformation which maps the three points z_1, z_2, z_3 uniquely onto w_1, w_2, w_3.

$$\begin{array}{rlcl} i) & z_1 = i, z_2 = 0, z_3 = 2i & \rightarrow & w_1 = -1, w_2 = 1, w_3 = 2, \\ ii) & z_1 = 1, z_2 = i, z_3 = 1+i & \rightarrow & w_1 = 1, w_2 = -1, w_3 \rightarrow \infty, \\ iii) & z_1 = 2, z_2 \rightarrow \infty, z_3 = i-1 & \rightarrow & w_1 = 2, w_2 = 3-i, w_3 \rightarrow \infty\ . \end{array}$$

3) Given are the linear fractional transformations

$$T_1(z) = \frac{z-1}{z+2}, \qquad T_2(z) = \frac{z}{z+2} \qquad \text{and} \qquad f_I(z) = \frac{1}{z}.$$

i) Compute the expressions

$$T_1 \circ T_2(z), \quad T_2 \circ T_1(z), \quad T_2 \circ f_I \circ T_2(z), \quad T_1 \circ f_I \circ T_1 \circ f_I(z) \ \text{ and } \ T_2^{-1} \circ T_1(z)$$

ii) Decompose your answers for $T_1 \circ T_2(z)$ and $T_2 \circ T_1(z)$ into a succession of translations f_T^Δ, rotations f_R^λ and the inversion map f_I.

4) i) Express

$$f_I \circ f_R^\lambda \circ f_I \circ f_R^\mu \qquad \text{for } \lambda, \mu \in \mathbb{C},$$

as a single rotation.

ii) Write

$$f_R^\lambda \circ f_T^\delta \circ f_I \circ f_T^\Delta \qquad \text{for } \lambda, \delta, \Delta \in \mathbb{C}$$

as a linear fractional transformations.

5) Compute the cross ratio (z_1, z_2, z_3, z_4) with $z_1 = 1 + \sqrt{3}$, $z_2 = 1 - 3i$, $z_3 = -1 - i$, $z_4 = 1 + i$ and decide whether the four points are situated on a circle, a line or neither of the two.

Solutions to exercises 2

1) Horizontal and vertical lines are mapped into hyperbolas:

$$z^2 = (x+iy)^2 = u + iv = x^2 - y^2 + i2xy$$

This means vertical lines $u = x^2 - y^2 = \kappa_1$ are rectangular hyperbolas and horizontal lines $v = 2xy = \kappa_2$ correspond to orthogonal hyperbolas.

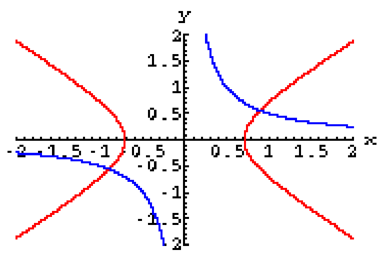

Figure 1: Hyperbolas for $\kappa_1 = 0.5$ and $\kappa_2 = 1$.

2) Solving the general formula

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

for $w = f(z)$, we find

$$f_1(z) = \frac{6i - 7z}{6i - 5z}, \qquad f_2(z) = \frac{iz}{1+i-z}, \qquad f_3(z) = \frac{(3-i)z}{1-i+z}.$$

3) i) We find

$$T_1 \circ T_2(z) = -\tfrac{2}{4+3z}, \qquad T_2 \circ T_1(z) = \tfrac{z-1}{3+3z}, \qquad T_2 \circ f_I \circ T_2(z) = \tfrac{2+z}{2+3z},$$
$$T_1 \circ f_I \circ T_1 \circ f_I(z) = z, \qquad T_2^{-1} \circ T_1(z) = \tfrac{2}{3}(z-1), \qquad \text{where } \; T_2^{-1}(z) = \tfrac{2z}{1-z}$$

ii)

$$\begin{aligned} T_1 \circ T_2(z) &= f_R^{-2} \circ f_I \circ f_T^4 \circ f_R^3(z) \\ T_2 \circ T_1(z) &= f_T^{1/3} \circ f_R^{-2} \circ f_I \circ f_T^3 \circ f_R^3(z) \end{aligned}$$

4) i)

$$f_I \circ f_R^\lambda \circ f_I \circ f_R^\mu = f_R^{\mu/\lambda}$$

ii)

$$f_R^\lambda \circ f_T^\delta \circ f_I \circ f_T^\Delta = \frac{\delta\lambda z + (\delta\lambda\Delta + \lambda)}{z + \Delta}$$

5) We find $T_c(x_4) = -1/\sqrt{3} \in \mathbb{R}$, which means the points must be on a line or a circle. Suppose the points are on a line $y = \alpha x + c$, then we compute for instance

$$\left.\begin{array}{ll} z_3: & -1 = -\alpha + c \\ z_4: & 1 = \alpha + c \end{array}\right\} \Rightarrow c = 0 \;, \Rightarrow \;\; z_3 : 1 = \alpha, \;\; \text{but } z_1 : 0 \neq (1+\sqrt{3}).$$

This means the points have to lie on a circle.