Mathematical Methods II

Exercises 3

1) Find a conformal map which maps the wedge region

$$\mathcal{W} = \{r, \theta : r \in \mathbb{R}^+, -\frac{\pi}{6} \leq \theta < \frac{\pi}{6}\}$$

in the z-plane onto the unit disc $|w| \leq 1$. Draw a figure and indicate the corresponding regions including some characteristic points.

2) Find a conformal map which maps the first quadrant in the z-plane

$$\mathcal{W}_2 = \{r, \theta : r \in \mathbb{R}^+, 0 \leq \theta < \frac{\pi}{2}\}$$

onto the unit disc $|w| \leq 1$. Draw a figure and indicate the corresponding regions including some characteristic points.

3) Find two different types of domains for the following function

$$f(z) = \ln\left(\frac{z+1}{z-1}\right)$$

such that it becomes single valued and analytic. In one case the domain should have one branch cut and in the other two. In each case compute the values

$$f(0 \pm i\varepsilon), \qquad f(2 \pm i\varepsilon) \qquad \text{and} \qquad f(-2 \pm i\varepsilon) \qquad \text{for} \quad \varepsilon \ll 1$$

and verify that the prescription

$$f(z) = \frac{1}{2}\left[f(z + i\varepsilon) + f(z - i\varepsilon)\right]$$

eliminates the ambiguities and gives the same answer in all cases.

Solutions to exercises 3

1) First we map the wedge onto the entire right half plane $\tilde{u} \geq 0$. The boundaries of the two regions must be mapped onto each other as the region inside stretches to infinity. This is achived by the map

$$\tilde{w} = \tilde{f}(z) = r^3 e^{3i\theta} = z^3, \quad \Rightarrow \arg z = \pm\frac{\pi}{6} \;\mapsto\; \arg\tilde{w} = \pm\frac{\pi}{2}.$$

In the next step we map the right half plane $\tilde{u} \geq 0$ onto the unit disc $|w| \leq 1$. Assuming this to be a linear fractional transformation, this map is determined by three points in the $\tilde{u}$-plane and three points in w-plane and a subsequent use of

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(\tilde{w} - \tilde{w}_1)(\tilde{w}_2 - \tilde{w}_3)}{(\tilde{w} - \tilde{w}_3)(\tilde{w}_2 - \tilde{w}_1)} \tag{1}$$

We choose three points on the imaginary axis $\tilde{w}_1 = 0, \tilde{w}_2 = i, \tilde{w}_3 \to i\infty$ and map them to $w_1 = -1, w_2 = i, w_3 = 1$. Therfore (1) yields

$$\frac{(w+1)(i-1)}{(w-1)(i+1)} = \frac{\tilde{w}}{i}.$$

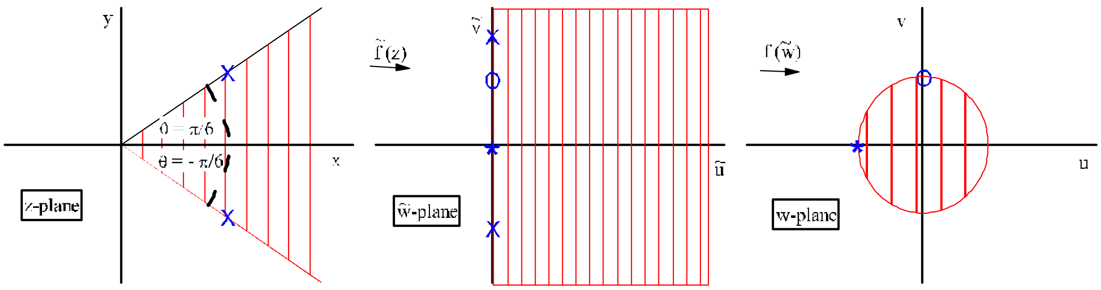
Solving this for w gives

$$w = f(\tilde{w}) = \frac{\tilde{w}-1}{\tilde{w}+1}$$

Thus the analytic function which maps the wedge $\mathcal{W}$ onto the unit circle is

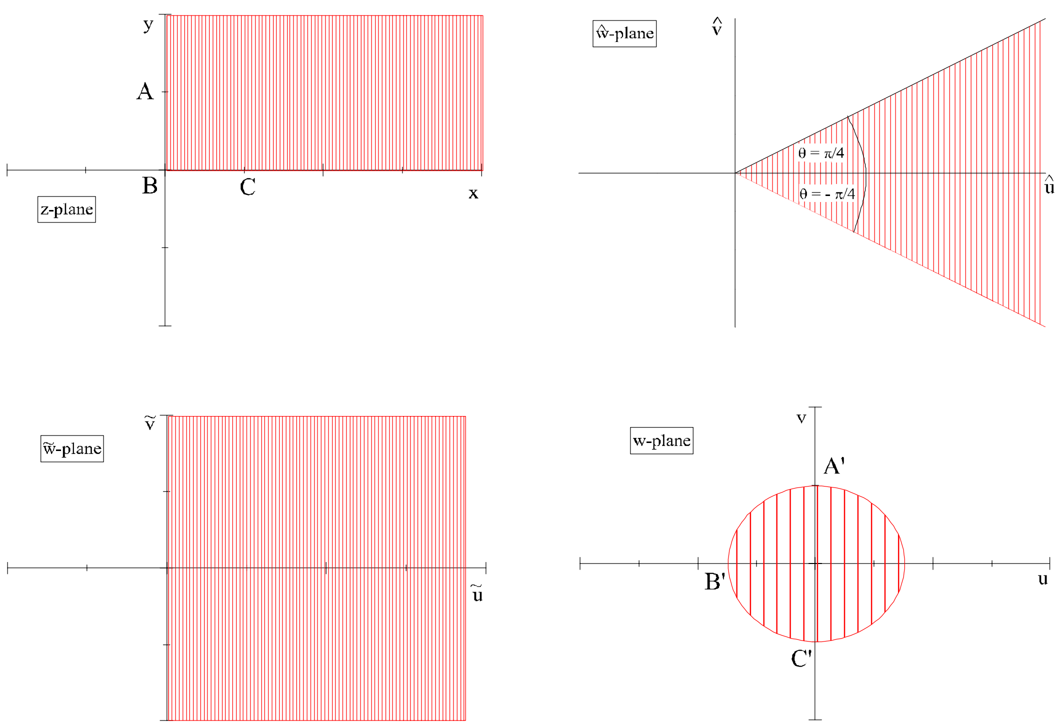
$$w = f \circ \tilde{f}(z) = \frac{z^3-1}{z^3+1}.$$

Include some characteristic points into the figure.



2) We start by rotating the first quadrant by $-i\pi/4$

$$\hat{w} = \hat{f}(z) = ze^{-i\pi/4}.$$



Now the problem is similar to the one in 1) only that the wedge is now

$$\mathcal{W}' = \{r, \theta : r \in \mathbb{R}^+, -\frac{\pi}{4} \leq \theta < \frac{\pi}{4}\}.$$

We proceed similarly as before and map this wedge to the right half plane

$$\tilde{w} = \tilde{f}(\hat{w}) = \hat{w}^2.$$

Finally we map the right half plane to the unit circle

$$w = f(\tilde{w}) = \frac{\tilde{w} - 1}{\tilde{w} + 1}.$$

Thus the map which maps the first quadrant onto the unit circle is

$$w = f \circ \tilde{f} \circ \hat{f}(z) = f \circ \tilde{f}(ze^{-i\pi/4}) = f(z^2 e^{-i\pi/2}) = f(-iz^2) = \frac{z^2 - i}{z^2 + i}.$$

Include some characteristic points into the figure.

3) The function $f(z)$ has two branch points at $z = 1$ and at $z = -1$. For the two arguments of the logarithm we can write

$$z + 1 = |z + 1|\, e^{i\theta_1} \qquad \text{and} \qquad z - 1 = |z - 1|\, e^{i\theta_2}$$

such that

$$f(z) = \ln\left(\frac{z+1}{z-1}\right) = \ln(z+1) - \ln(z-1) = \ln\left|\frac{z+1}{z-1}\right| + i(\theta_1 - \theta_2)$$

We have now various choices for the restriction on θ_1 and θ_2 :

i) Assume the principal values for the logarithms:

$$-\pi < \theta_1 \leq \pi \quad \text{and} \quad -\pi < \theta_2 \leq \pi$$

Let us now consider the different regions on the real axis:

- $z \in (1, \infty)$: On this part of the axis there is no problem as both θ_1 and θ_2 are both continuous when crossing the axis.
- $z \in (-1, 1)$: On this line segment θ_1 is continuous, but θ_2 jumps and therefore we require a cut.
- $z \in (-\infty, 1)$: When crossing this part of the axis both θ_1 and θ_2 are discontinuous. However, the relevant quantity, which is the difference $\theta_1 - \theta_2$ is continuous. Above the axis we have $\theta_1 = \theta_2 = \pi$, such that $\theta_1 - \theta_2 = 0$ and below the axis we have $\theta_1 = \theta_2 = -\pi$ and therefore also $\theta_1 - \theta_2 = 0$.

This means we only need a branch cut at the line segment $(-1, 1)$ in order to make this function single valued and analytic.

$$f(-2 \pm i\varepsilon) \approx \ln 1/3, \qquad f(2 \pm i\varepsilon) \approx \ln 3/2 \qquad f(\pm i\varepsilon) \approx \mp i\pi$$

ii) Next we assume:

$$0 < \theta_1 \leq 2\pi \qquad \text{and} \qquad 0 < \theta_2 \leq 2\pi$$

Let us consider again the different regions on the real axis:

- $z \in (1, \infty)$: When crossing this part of the axis both θ_1 and θ_2 are discontinuous, but with the same argument as before the difference $\theta_1 - \theta_2$ is continuous. Above the axis we have $\theta_1 = \theta_2 = 0$, such that $\theta_1 - \theta_2 = 0$ and below the axis we have $\theta_1 = \theta_2 = 2\pi$ and therefore also $\theta_1 - \theta_2 = 0$.
- $z \in (-1, 1)$: On this line segment θ_2 is continuous, but θ_1 jumps and therefore we require a cut.
- $z \in (-\infty, 1)$: On this part of the axis there is no problem as both θ_1 and θ_2 are continuous when crossing the axis.

This means once again we only need a branch cut at the line segment $(-1, 1)$ in order to make this function single valued and analytic.

$$f(-2 \pm i\varepsilon) \approx \ln 1/3, \qquad f(2 \pm i\varepsilon) \approx \ln 3/2 \qquad f(\pm i\varepsilon) \approx \mp i\pi$$

iii) Next we assume:

$$-\pi < \theta_1 \leq \pi \qquad \text{and} \qquad 0 < \theta_2 \leq 2\pi$$

Let us consider again the different regions on the real axis:

- $z \in (1, \infty)$: On this part of the axis θ_1 is continuous, but θ_2 jumps and therefore we require a cut.
- $z \in (-1, 1)$: On this line segment there is no problem as both θ_1 and θ_2 are continuous when crossing the axis.
- $z \in (-\infty, 1)$: On this part of the axis θ_2 is continuous, but θ_1 jumps and therefore we require another cut.

This means in this case we need two branch cut one at $(-\infty, 1)$ and the other at $(1, \infty)$ in order to make this function single valued and analytic.

$$f(-2 \pm i\varepsilon) \approx \ln 1/3 \pm i\pi, \qquad f(2 \pm i\varepsilon) \approx \ln 3/2 \pm i\pi \qquad f(\pm i\varepsilon) \approx 0$$

When averaging across the cut we obtain in all three cases

$$f(-2 \pm i\varepsilon) \approx \ln 1/3, \qquad f(2 \pm i\varepsilon) \approx \ln 3/2 \qquad f(\pm i\varepsilon) \approx 0.$$