Mathematical Methods II

Exercises 4

1) (Recap from calculus 2) Show that the Laplace equation in Cartesian coordinates

$$\Delta\phi(x, y) = 0$$

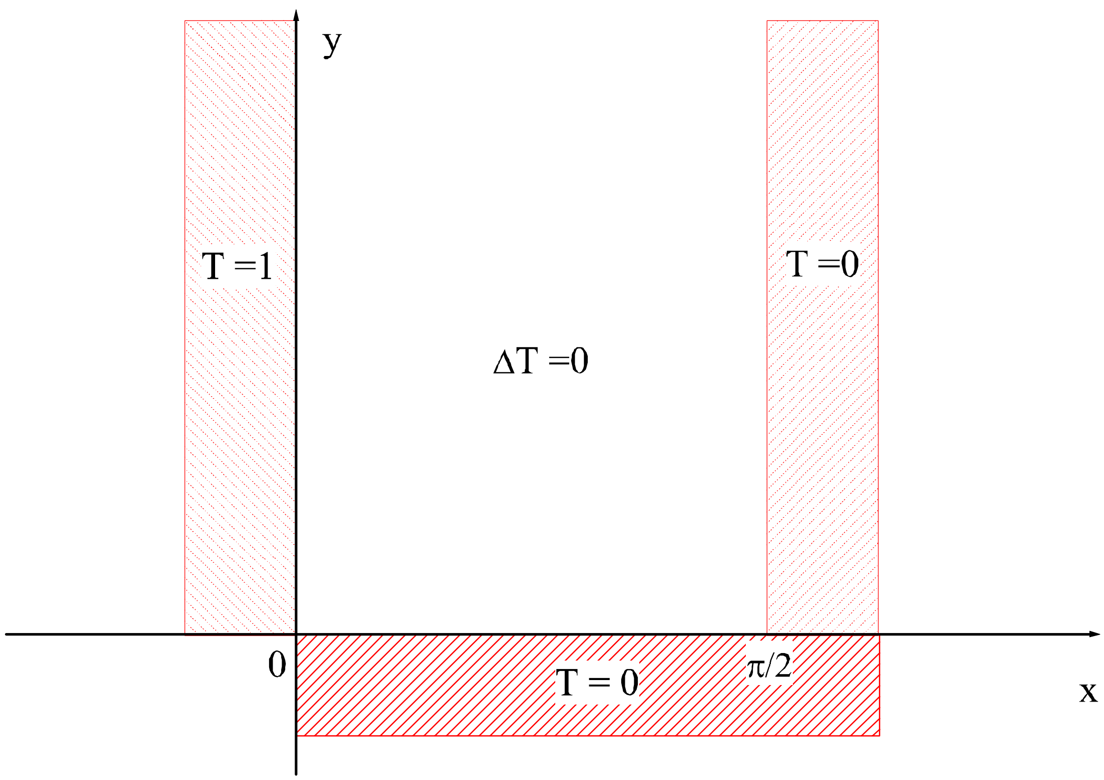
transforms into

$$\Delta\phi(r, \vartheta) = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0$$

when using polar coordinates $x = r\cos\vartheta, y = r\sin\vartheta$. This was used in section 2.3 of the lecture.

2) Find the potential function for the entire xy-plane when two infinitely long cylinders $|z| = 1$ and $|z - x_0| = x_0$ are non-coaxial. Place the cylinders at the constant potentials $\phi_1 = 0$ at $|z| = 1$ and $\phi_0 = 220V$ at $|z - x_0| = x_0$. Take the value of the center of the smaller cylinder and its radius to be i) $x_0 = 2/5$ and ii) $x_0 = 4/17$.

3) Find the steady state temperature function in the semi-infinite strip as depicted in the figure.



This means solve the Dirichlet problem

$$\Delta T = 0, \quad T(\pi/2, y) = 0, \quad T(0, y) = 1, \qquad T(x, 0) = 0, \qquad \text{for } 0 < x < \frac{\pi}{2}, y > 0.$$

Solutions to exercises 4

1) Use the chain rule to compute

$$\frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta \quad (1)$$

$$\frac{\partial\phi}{\partial\vartheta} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\vartheta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\vartheta} = -\frac{\partial\phi}{\partial x}r\sin\vartheta + \frac{\partial\phi}{\partial y}r\cos\vartheta \quad (2)$$

Solving for $\partial\phi/\partial x$ and $\partial\phi/\partial y$

$$(1)r\cos\vartheta - (2)\sin\vartheta \ : \ \frac{\partial\phi}{\partial r}r\cos\vartheta - \frac{\partial\phi}{\partial\vartheta}\sin\vartheta = \frac{\partial\phi}{\partial x}r(\cos^2\vartheta + \sin^2\vartheta)$$

$$(1)r\cos\vartheta - (2)\sin\vartheta \ : \ \frac{\partial\phi}{\partial r}r\sin\vartheta + \frac{\partial\phi}{\partial\vartheta}\cos\vartheta = \frac{\partial\phi}{\partial y}r(\cos^2\vartheta + \sin^2\vartheta)$$

therefore

$$\frac{\partial\phi}{\partial x} = \cos\vartheta\frac{\partial\phi}{\partial r} - \frac{1}{r}\sin\vartheta\frac{\partial\phi}{\partial\vartheta}$$

$$\frac{\partial\phi}{\partial y} = \sin\vartheta\frac{\partial\phi}{\partial r} + \frac{1}{r}\cos\vartheta\frac{\partial\phi}{\partial\vartheta}.$$

Next

$$\frac{\partial^2\phi}{\partial x^2} = \cos\vartheta\frac{\partial}{\partial r}\left(\cos\vartheta\frac{\partial\phi}{\partial r} - \frac{1}{r}\sin\vartheta\frac{\partial\phi}{\partial\vartheta}\right) - \frac{1}{r}\sin\vartheta\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\frac{\partial\phi}{\partial r} - \frac{1}{r}\sin\vartheta\frac{\partial\phi}{\partial\vartheta}\right)$$

$$\frac{\partial^2\phi}{\partial y^2} = \sin\vartheta\frac{\partial}{\partial r}\left(\sin\vartheta\frac{\partial\phi}{\partial r} + \frac{1}{r}\cos\vartheta\frac{\partial\phi}{\partial\vartheta}\right) + \frac{1}{r}\cos\vartheta\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\phi}{\partial r} + \frac{1}{r}\cos\vartheta\frac{\partial\phi}{\partial\vartheta}\right)$$

Adding these equations gives

$$\Delta\phi(r,\vartheta) = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0.$$

2)

$$i) \quad x_0 = \frac{2}{5} \quad \Rightarrow r_0 = \frac{1}{2} \quad \Rightarrow \phi(r) = -220\frac{\ln\left|\frac{2z-1}{z-2}\right|}{\ln 2}V$$

$$ii) \quad x_0 = \frac{4}{17} \quad \Rightarrow r_0 = \frac{1}{4} \quad \Rightarrow \phi(r) = -220\frac{\ln\left|\frac{4z-1}{z-2}\right|}{\ln 4}V$$

3) The solution is

$$T(x,y) = \frac{2}{\pi}\arctan\left(\frac{\tanh y}{\tan x}\right)$$