Mathematical Methods II

Exercises 5

1) Compute the Fourier transforms $\mathcal{F}u(x)$ of the following functions

i)

$$u(x) = \begin{cases} -1 & \text{for } -1 \leq x \leq 0 \\ 1 & \text{for } \quad 0 < x \leq 1 \\ 0 & \text{for } \quad |x| > 1 \end{cases}$$

ii)

$$u(x) = \begin{cases} x & \text{for } |x| \leq \lambda \\ 0 & \text{for } |x| > \lambda \end{cases}$$

iii)

$$u(x) = (4x^2 - 2)e^{-x^2}$$

Exploit the fact that $\mathcal{F}u'(x) = ix\mathcal{F}u(x)$ and $\mathcal{F}(e^{-x^2}) = \sqrt{\pi}e^{-x^2/4}$.

iv)

$$u(x) = x^2 e^{-\lambda x^2}$$

Exploit the scaling property of the Fourier transform $\mathcal{F}u(\lambda x) = ix\mathcal{F}u(x)$ and $\mathcal{F}(e^{-x^2}) = \sqrt{\pi}e^{-x^2/4}$.

2) By using a similar technique as for the heat equation in the lecture, solve the wave equation

$$\frac{d^2\phi(x,t)}{dt^2} - \frac{d^2\phi(x,t)}{dx^2} = 0$$

subject to the initial condition $\phi(x,0) = 0$ and boundary condition $\partial_t\phi(x,t) = f(x)$.

3) The function $\psi(x,y)$ satisfies the Laplace equation

$$\frac{d^2\psi(x,y)}{dx^2} + \frac{d^2\psi(x,y)}{dy^2} = 0 \quad \text{for } -\infty < x < \infty, y > 0$$

subject to the boundary condition $\psi(x,0) = f(x)$, $\lim_{y\to\infty}\psi(x,y) = 0$. Solve this Dirichlet problem by using Fourier transforms.

4) The heat equation in two space dimensions is

$$\Delta T(x,y,t) = \partial_t T(x,y,t).$$

Write this equation in polar coordinates and assume that $T(x,y,t)$ as a function of r, ϑ and t factorises as

$$T(r,\vartheta,t) = R(r)\Theta(\vartheta)\tau(t).$$

Find the differential equation satisfied by $R(r), \Theta(\vartheta)$ and $\tau(t)$.

Solutions to exercises 5

1) i)

$$\mathcal{F}u(x) = \frac{2i}{x}\left[\cos(x) - 1\right]$$

ii)

$$\mathcal{F}u(x) = \frac{2i}{x}\left[\lambda\cos(x\lambda) - \frac{\sin(x\lambda)}{x}\right]$$

iii)

$$\mathcal{F}u(x) = (ix)^2\mathcal{F}(e^{-x^2}) = -x^2\sqrt{\pi}e^{-x^2/4}$$

iv)

$$\mathcal{F}u(x) = -\frac{d}{d\lambda}\mathcal{F}(e^{-\lambda x^2}) = -\frac{d}{d\lambda}\left(\sqrt{\frac{\pi}{\lambda}}e^{-x^2/4\lambda}\right) = \frac{e^{-x^2/4\lambda}}{4}\sqrt{\frac{\pi}{\lambda^5}}(2\lambda - x^2)$$

2)

$$\phi(x,t) = \frac{1}{2}\int_{-t}^{t} f(x-s)ds$$

3)

$$\psi(x,y) = \int_{-\infty}^{\infty} g(s)e^{-sy}e^{-isx}ds \quad \text{where} \quad \mathcal{F}g(x) = f(x)$$

4)

$$\begin{aligned} \tau' + \alpha^2\tau &= 0 \\ \Theta'' + \beta^2\Theta &= 0 \\ r^2R'' + rR' + (\alpha^2r^2 - \beta^2)R &= 0 \end{aligned}$$

2) $$\partial_t^2 \phi(x,t) - \partial_x^2 \phi(x,t) = 0$$

define $\phi(x,t) = \mathcal{F}_x u(x,t) = \int_{-\infty}^{+\infty} u(s,t)\, e^{-isx}\, ds$

$$\Rightarrow \partial_x^2 \phi(x,t) = \int_{-\infty}^{+\infty} u(x,t)(-s^2)\, e^{-isx}\, ds = \int_{-\infty}^{+\infty} \partial_t^2 u(x,t)\, e^{-isx}\, ds$$

$$\Rightarrow \partial_t^2 u(s,t) + s^2 u(s,t) = 0 \Rightarrow u(s,t) = c_1(s) e^{ist} + c_2(s) e^{-ist}$$

$$\Rightarrow \phi(x,t) = \int_{-\infty}^{+\infty} \left[c_1(s)\, e^{-is(x-t)} + c_2(s)\, e^{-is(x+t)}\right] ds$$

initial value $\Rightarrow$ $\phi(x,0) = 0 = \int_{-\infty}^{+\infty} (c_1(s) + c_2(s))\, e^{-isx}\, ds \Rightarrow c_1(s) = -c_2(s) = c(s)$

define $\tilde{c}(x) = \int_{-\infty}^{+\infty} c(s)\, e^{-isx}\, ds$

$$\Rightarrow \phi(x,t) = \tilde{c}(x-t) - \tilde{c}(x+t) = -\int_{-t}^{t} \partial_s \tilde{c}(x-s)\, ds$$

boundary condition $\Rightarrow$ $\partial_t \phi(x,t)\big|_{t=0} = f(x) = \partial_t \tilde{c}(x-t) - \partial_t \tilde{c}(x+t)\big|_{t=0}$

$$= -\partial_s \tilde{c}(s) - \partial_s \tilde{c}(s) = -2\partial_s \tilde{c}(s)$$

$$\Rightarrow \partial_s \tilde{c}(s) = -\tfrac{1}{2} f(s) \Rightarrow \phi(x,t) = \tfrac{1}{2}\int_{-t}^{t} ds\, f(x-s)$$

3) $$\partial_x^2 \psi(x,y) + \partial_y^2 \psi(x,y) = 0$$

define $\psi(x,y) = \mathcal{F}_x u(x,y) = \int_{-\infty}^{+\infty} u(s,y)\, e^{-isx}\, ds$

$$\Rightarrow \partial_x^2 \psi(x,y) = \int u(s,y)(-s^2)\, e^{-isx}\, ds = -\int \partial_y^2 u(s,y)\, e^{-isy}\, ds$$

$$\Rightarrow \partial_y^2 u(s,y) - s^2 u(s,y) = 0 \Rightarrow u(s,y) = c_1(s) e^{sy} + c_2(s) e^{-sy}$$

boundary condition $\Rightarrow$ $\lim_{y\to\infty} \psi(x,y) = 0 \Rightarrow \lim_{y\to\infty} u(x,y) = 0 \Rightarrow c_1(s) = 0$

$$\Rightarrow \psi(x,y) = \int c_2(s)\, e^{-sy}\, e^{-isx}\, ds$$

initial condition $\Rightarrow$ $\psi(x,0) = \int c_2(s)\, e^{-isx}\, ds = f(x) = \mathcal{F} g(x)$

$$\Rightarrow c_2(s) = g(s)$$

$$\Rightarrow \psi(x,y) = \int g(s)\, e^{-sy}\, e^{-isx}\, ds \quad \text{with } \mathcal{F} g(x) = f(x)$$