Mathematical Methods II

Exercises 6

1) Assume for the following functions that $u(x) = 0$ for $x < 0$ and $\lambda, \mu \in \mathbb{R}^+$.

$(i)\ u(x) = \sinh \lambda x$ $\quad (ii)\ u(x) = \cosh \lambda x$ $\quad (iii)\ u(x) = x \cosh \lambda x$

$(iv)\ u(x) = x \sin \lambda x$ $\quad (v)\ u(x) = e^{\mu x} \sin \lambda x$ $\quad (vi)\ u(x) = \sinh \lambda x \sin \lambda x$

$(vii)\ u(x) = \sinh \lambda x - \sin \lambda x$ $\quad (viii)\ u(x) = \frac{1}{\sqrt{\lambda x}}$ $\quad (ix)\ u(x) = \cosh \lambda x - \cos \lambda x$

Compute the Laplace transforms $\mathcal{L}u(x)$ for $u(x)$.

2) Assume that $\lambda, \mu \in \mathbb{R}^+$ with $\lambda \neq \mu$. Compute the inverse Laplace transforms $\mathcal{L}^{-1}v(x)$ for the following functions

$(i)\ v(x) = \frac{1}{(x+\lambda)(x+\mu)}$ $\quad (ii)\ v(x) = \frac{1}{x^2-4x-3}$ $\quad (iii)\ v(x) = \frac{2+3x}{8+6x+x^2}$

$(iv)\ v(x) = \frac{e^{-4x}}{x+3}$ $\quad (v)\ v(x) = \frac{xe^{-x}}{x+1}$ $\quad (vi)\ v(x) = \frac{3x^2-2}{(x^2+6)^2}$

You may eithe refer to Laplace transforms previously computed or use the Bromwich integral representation for the inverse Laplace transform..

3) By means of Laplace transforms solve the second order differential equation

$$u''(x) + 2u'(x) + 2u(x) = \delta(x - \pi)$$

subject to the boundary conditions $u'(0) = u(0) = 0$.

4) By means of Laplace transforms solve the second order differential equation

$$u''(x) + 4u(x) = H(x) - H(x - \pi)$$

subject to the boundary conditions

$$i)\ \ u(0) = 1, u'(0) = 0 \qquad \text{and} \qquad ii)\ \ u(0) = 0, u'(0) = 1.$$

5) By means of Laplace transforms solve the heat equation

$$\phi_t(x,t) - \kappa \phi_{xx}(x,t) = 0 \qquad \text{for } x > 0, t > 0,$$

subject to the boundary conditions

$$\phi(x,0) = 0,\ \ \phi_x(0,t) = -\lambda/\kappa \qquad \text{and} \qquad \lim_{x\to\infty} \phi(x,t) = 0.$$

Leave your answer in form of an integral representation.

Solutions to exercises 6

1)

$(i)\ \mathcal{L}u(x) = \frac{\lambda}{x^2-\lambda^2}$ $(ii)\ \mathcal{L}u(x) = \frac{x}{x^2-\lambda^2}$ $(iii)\ \mathcal{L}u(x) = \frac{x^2+\lambda^2}{(x^2-\lambda^2)^2}$

$(iv)\ \mathcal{L}u(x) = \frac{2\lambda x}{(x^2+\lambda^2)^2}$ $(v)\ \mathcal{L}u(x) = \frac{\lambda}{\lambda^2+(x-\mu)^2}$ $(vi)\ \mathcal{L}u(x) = \frac{2x\lambda^2}{x^4+4\lambda^4}$

$(vii)\ \mathcal{L}u(x) = \frac{2\lambda^3}{x^4-\lambda^4}$ $(viii)\ \mathcal{L}u(x) = \frac{\sqrt{\pi}}{\sqrt{\lambda x}}$ $(ix)\ \mathcal{L}u(x) = \frac{2x\lambda^2}{x^4-\lambda^4}$

2)

$(i)\ \mathcal{L}^{-1}v(x) = \frac{e^{-\lambda x}-e^{-\mu x}}{\mu-\lambda}$ $(ii)\ \mathcal{L}^{-1}v(x) = \frac{1}{2}\left(e^{3x}-e^{x}\right)$

$(vi)\ \mathcal{L}^{-1}v(x) = \frac{5}{3}\ x\cos(x\sqrt{6}) + \left(\frac{2}{3}\right)^{3/2}\ \sin(x\sqrt{6})$ $(iv)\mathcal{L}^{-1}v(x) = \ H(x-4)e^{12-3x}$

$(v)\ \mathcal{L}^{-1}v(x) = \ H(x-1)\left[\delta(x-1) - \frac{1}{2}e^{1-x}\right]$ $(iii)\ \mathcal{L}^{-1}v(x) = 5e^{-4x} - 2e^{-2x}$

3)

$$u(x) = -H(x-\pi)e^{\pi-x}\sin x$$

4)

$$\begin{aligned} i)\ \ u(x) &= \cos 2x + \frac{1}{2}\sin^2 x\,[H(x) - H(x-\pi)] \\ ii)\ \ u(x) &= \frac{1}{2}\sin 2x + \frac{1}{2}\sin^2 x\,[H(x) - H(x-\pi)] \end{aligned}$$

5)

$$\phi(x,t) = \frac{\lambda}{\sqrt{\pi\kappa}}\int_0^\infty H(t-s)\frac{e^{-x^2/4\kappa s}}{\sqrt{s}}ds$$