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(q_1,q_2)-DEFORMED SUSY ALGEBRA FOR $SU_{q_1/q_2}(n)$ -INVARIANT BOSONS*

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PLAN

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1. Quantum Group Invariant Two Parameter Deformed Boson Algebra

$$
\begin{aligned}
a_1 a_2 &= q_1 q_2^{-1} a_2 a_1, \\
a_1^* a_2^* &= q_2 q_1^{-1} a_2^* a_1^*, \\
a_1 a_2^* &= q_1 q_2 a_2^* a_1, \\
a_1 a_1^* - q_1^2 a_1^* a_1 &= q_2^{2N_B}, \\
a_2 a_2^* - q_1^2 a_2^* a_2 &= a_1 a_1^* - q_2^2 a_1^* a_1, \\
q_1^{2N_B} &= a_2 a_2^* - q_2^2 a_2^* a_2,
\end{aligned}
\qquad (1)
$$

(M. Arik et al., Z. Phys. C. 55 (1992) 89)

- where $q_1 \neq q_2$, $(q_1, q_2) \in R^+$, and $N_B = N_1 + N_2$.

$$a_1^* a_1 + a_2^* a_2 = [N_1 + N_2] = [N_B], \quad (2)$$

- whose spectrum is given by

$$[n] = \frac{q_2^{2n} - q_1^{2n}}{q_2^2 - q_1^2}, \quad (3)$$

- which is called the generalized Fibonacci basic integer.
- $q_1 \leftrightarrow q_2$ symmetry !
- This (q_1,q_2)- deformed bosonic algebra is invariant under the quantum group $SU_r(2)$ with $r = q_1/q_2$. How?

- Our system is invariant under the following transformation:

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = T \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \alpha & r\beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad (4)$$

such that T$\in SU_r(2)$ with $r = q_1/q_2$, and

$$\alpha\beta = r\beta\alpha, \qquad \alpha\beta^* = r\beta^*\alpha,$$

$$\beta\beta^* = \beta^*\beta, \qquad \alpha^*\alpha + \beta\beta^* = 1, \qquad (5)$$

$$\alpha\alpha^* + r^2\beta^*\beta = 1.$$

(Note that in this computation, the matrix elements of T are assumed to commute with a_1, a_2, a_1^*, a_2^*.)

$SU_{q_1/q_2}(n)$- invariant two-parameter deformed boson algebra:

$$a_i a_k = q_1 q_2^{-1} a_k a_i, \qquad i<k,$$

$$a_i^* a_k^* = q_2 q_1^{-1} a_k^* a_i^*, \qquad i<k,$$

$$a_i a_k^* = q_1 q_2 a_k^* a_i, \qquad i \neq k, \tag{6}$$

$$a_1 a_1^* - q_1^2 a_1^* a_1 = q_2^{2N_B},$$

$$a_k a_k^* - q_1^2 a_k^* a_k = a_{k-1} a_{k-1}^* - q_2^2 a_{k-1}^* a_{k-1}, \qquad k=2,\ldots,n,$$

$$q_1^{2N_B} = a_n a_n^* - q_2^2 a_n^* a_n,$$

$$a_1^* a_1 + a_2^* a_2 + \ldots\ldots + a_n^* a_n = [N_1 + \ldots\ldots + N_n] = [N_B], \tag{7}$$

whose spectrum is given by the Fibonacci basic numbers [n] in Eq. (3).

(M.Arik et al., Z. Phys. C 55 (1992) 89)

$SU_{q_1/q_2}(n)$-invariant (q_1, q_2)-deformed bosonic oscillator algebra

$q_1=q_2=1$

Undeformed boson algebra

$q_2=1$

The one-parameter deformed boson algebra invariant under the $SU_{q_1}(n)$.

(W. Pusz and S.L. Woronowicz, Rep. Math. Phys. 27 (1989) 231.)

$q_1=q_2=q$

The q-deformed bosonic Newton oscillator algebra invariant under the $SU(n)$.

(M.Arik et.al. J. Phys. A 32 (1999) L371)

$SU_r(n)$-invariant

(q_1, q_2)-deformed

bosons with $r = q_1/q_2$.

q- bosons

$$c_i c_i^* - q^{-1} c_i^* c_i = q^N,$$

$$[c_i, c_j^*] = 0 = [c_i, c_j],$$

$$N|n\rangle = n|n\rangle,$$

$$J_+ = c_2^* c_1, \quad J_- = c_1^* c_2,$$

$$2J_3 = N_2 - N_1,$$

$$[J_3, J_\pm] = \pm J_\pm,$$

$$[J_+, J_-] = [2J_3],$$

where

$$[\times] = \frac{q^{\times} - q^{-\times}}{q - q^{-1}}$$

(A. J. Macfarlane, J. Phys. A 22 (1989) 4581; L.C. Biedenharn, J. Phys. A 22 (1989) L873; Y. J. Ng, J. Phys. A 23 (1990) 1023)

2. (q_1, q_2)-deformed SUSY algebra

Two (q_1,q_2)-deformed bosons

with $SU_{q_1/q_2}(2)$-symmetry

$$a_1, a_1^*, a_2, a_2^*$$

Two undeformed fermions

$$f_1,\ f_1^*,\ f_2,\ f_2^*.$$

$$\{f_i, f_j^*\} = \delta_{ij}, \qquad i, j = 1,2,$$

$$\{f_i, f_j\} = 0, \qquad \{f_i^*, f_j^*\} = 0,$$

$$f_i^2 = (f_i^*)^2 = 0,$$

.

$$Q_1 = a_1^* f_1, \qquad Q_2 = a_2^* f_2,$$

$$Q_1^* = f_1^* a_1, \qquad Q_2^* = f_2^* a_2, \qquad (8)$$

$$Q_1^2 = Q_2^2 = (Q_1^*)^2 = (Q_2^*)^2 = 0.$$

$$a_i f_m = f_m a_i,$$

$$a_i f_m^* = f_m^* a_i, \qquad i, m = 1,2.$$

$$\{Q_1, Q_2\}_{(q_1^2+q_2^2)/2q_1q_2} = 0, \quad \{Q_1, Q_2^*\}_{q_1^{-1}q_2^{-1}} = 0,$$

$$\{Q_1, Q_1^*\}_{2/(q_1^2+q_2^2)} = H_1 = a_1^* a_1 + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2} \right) f_1^* f_1$$

$$\{Q_2, Q_2^*\}_{2/(q_1^2+q_2^2)} = H_2 = a_2^* a_2 + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2} \right) f_2^* f_2 \qquad (9)$$

$$[H_1, Q_1]_{(q_1^2+q_2^2)/2} = 0, \qquad [H_2, Q_2]_{(q_1^2+q_2^2)/2} = 0,$$

$$[H_1, Q_2]_{(q_1^2+q_2^2)/2} = 0, \qquad [H_2, Q_1]_{(q_1^2+q_2^2)/2} = 0,$$

where $\{A, B\}_x = AB + xBA$ $\qquad [A, B]_x = AB - xBA$

$$H = H_1 + H_2 = (a_1^* a_1 + a_2^* a_2) + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2} \right) (f_1^* f_1 + f_2^* f_2), \quad (10)$$

where $N_B = N_1 + N_2$.

- In the limit $q_2 = q_1 = 1$, $H \longrightarrow$ the standard SUSY Hamiltonian for two undeformed bosons and two undeformed fermions.

(E. Witten, Nucl. Phys. B 185 (1981) 513; M. de Crombrugghe and Rittenberg, Ann. Phys. 151(1983) 99.)

- For the system containing *n*-deformed bosons and *n*-undeformed fermions,

  The *2n* supercharges:

$$Q_i = a_i^* f_i, \qquad Q_i^* = f_i^* a_i, \tag{11}$$

$$\{Q_i, Q_j\}_{(q_1^2+q_2^2)/2q_1q_2} = 0, \qquad i < j,$$

$$\{Q_i, Q_j^*\}_{q_1^{-1}q_2^{-1}} = 0,$$

$$[H_i, Q_j]_{(q_1^2+q_2^2)/2} = 0, \qquad i \geq j,$$

$$[H_i, Q_j]_{(q_1^2+q_2^2)/2} = 0, \qquad i < j,$$

$$\{Q_i, Q_i^*\}_{2/(q_1^2+q_2^2)} = H_i = a_i^* a_i + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2} \right) f_i^* f_i \tag{12}$$

where N_B is the total boson number operator.

$$H = \sum_{i=1}^{n} H_i = \sum_{i=1}^{n} a_i^* a_i + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2} \right) \sum_{i=1}^{n} f_i^* f_i. \tag{13}$$

3. Fock space representation of the (q_1, q_2)-deformed SUSY algebra

- For the bosonic sector (a_1, a_1^*, a_2, a_2^*):

$|n_1, n_2\rangle$, where $n_1, n_2 = 0,1,2,....$ $\quad a_i|0,0\rangle = 0, \quad i = 1,2.$

$$a_1|n_1,n_2\rangle = q_2^{n_2}\sqrt{[n_1]}|n_1-1,n_2\rangle, \qquad a_1^* a_1|n_1,n_2\rangle = q_2^{2n_2}[n_1]|n_1,n_2\rangle, \quad (14)$$

$$a_1^*|n_1,n_2\rangle = q_2^{n_2}\sqrt{[n_1+1]}|n_1+1,n_2\rangle, \qquad a_2^* a_2|n_1,n_2\rangle = q_1^{n_1}[n_2]|n_1,n_2\rangle,$$

$$a_2|n_1,n_2\rangle = q_1^{n_1}\sqrt{[n_2]}|n_1,n_2-1\rangle,$$

$$a_2^*|n_1,n_2\rangle = q_1^{n_1}\sqrt{[n_2+1]}|n_1,n_2+1\rangle,$$

where $[n_i]$ is the Fibonacci basic number in Eq.(3).

$$|n_1,n_2\rangle = \frac{1}{\sqrt{[n_1]![n_2]!}}(a_2^*)^{n_2}(a_1^*)^{n_1}|0,0\rangle. \quad (15)$$

- For the fermionic sector (f_1, f_1^*, f_2, f_2^*):

$$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle.$$

$$f_1|\uparrow\uparrow\rangle = |\downarrow\uparrow\rangle, \quad f_2|\uparrow\uparrow\rangle = |\uparrow\downarrow\rangle, \quad f_1^*|\uparrow\uparrow\rangle = 0, \quad f_2^*|\uparrow\uparrow\rangle = 0,$$

$$f_1|\downarrow\downarrow\rangle = 0, \quad f_2|\downarrow\downarrow\rangle = 0, \quad f_1^*|\downarrow\downarrow\rangle = |\uparrow\downarrow\rangle, \quad f_2^*|\downarrow\uparrow\rangle = 0,$$

$$f_1|\uparrow\downarrow\rangle = |\downarrow\downarrow\rangle, \quad f_2|\uparrow\downarrow\rangle = 0, \quad f_1^*|\uparrow\downarrow\rangle = 0, \quad f_2^*|\uparrow\downarrow\rangle = |\uparrow\uparrow\rangle, \tag{16}$$

$$f_1|\downarrow\uparrow\rangle = 0, \quad f_2|\downarrow\uparrow\rangle = |\downarrow\downarrow\rangle, \quad f_1^*|\downarrow\uparrow\rangle = |\uparrow\uparrow\rangle, \quad f_2^*|\downarrow\downarrow\rangle = |\downarrow\uparrow\rangle,$$

$$M_1|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle, \quad M_2|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle, \quad M_1|\downarrow\uparrow\rangle = 0, \quad M_2|\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle,$$

$$M_1|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle, \quad M_2|\uparrow\downarrow\rangle = 0, \quad M_1|\downarrow\downarrow\rangle = 0, \quad M_2|\downarrow\downarrow\rangle = 0.$$

$$\left|(n_1,n_2);\uparrow\uparrow\right\rangle,\quad \left|(n_1,n_2);\uparrow\downarrow\right\rangle,\quad \left|(n_1,n_2);\downarrow\uparrow\right\rangle,\quad \left|(n_1,n_2);\downarrow\downarrow\right\rangle.$$

$$\left(H = H_1 + H_2 = \sum_{i=1}^{2} a_i^* a_i + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2} \right) \sum_{i=1}^{2} f_i^* f_i, \right)$$

$$H\left|(n_1,n_2);\uparrow\uparrow\right\rangle = \left\{ [n_1+n_2] + 2\left(\frac{q_1^{2(n_1+n_2)} + q_2^{2(n_1+n_2)}}{q_1^2+q_2^2} \right) \right\} \left|(n_1,n_2);\uparrow\uparrow\right\rangle,$$

$$H\left|(n_1,n_2);\uparrow\downarrow\right\rangle = \left\{ [n_1+n_2] + \left(\frac{q_1^{2(n_1+n_2)} + q_2^{2(n_1+n_2)}}{q_1^2+q_2^2} \right) \right\} \left|(n_1,n_2);\uparrow\downarrow\right\rangle,$$

$$H\left|(n_1,n_2);\downarrow\uparrow\right\rangle = \left\{ [n_1+n_2] + \left(\frac{q_1^{2(n_1+n_2)} + q_2^{2(n_1+n_2)}}{q_1^2+q_2^2} \right) \right\} \left|(n_1,n_2);\downarrow\uparrow\right\rangle,$$

$$H\left|(n_1,n_2);\downarrow\downarrow\right\rangle = [n_1+n_2] \left|(n_1,n_2);\downarrow\downarrow\right\rangle,$$

$$\left|(0,0);\downarrow\downarrow\right\rangle \Rightarrow \quad E = 0, \qquad \left\{ \begin{array}{l} \left|(0,1);\downarrow\downarrow\right\rangle \\ \left|(1,0);\downarrow\downarrow\right\rangle \end{array} \right\} \Rightarrow \quad E = 1$$

4. Concluding remarks

The remarkable differences between our construction* and the earlier *q*-deformed SUSY algebra constructions:

(Kulish and Reshetikhin, 1989; Chaichian, and Kulish, 1990; Chaichian, et al., 1991; Floreanini et al.,1990; Parthasarathy and Viswanathan,1991; Spiridonov, 1992; Chung,1995; Isaev, and Malik, 1992; Hegazi and Mansour, 2001; Algin, 2002; F. Besnard, 2004)

- (q_1,q_2)-deformed bosonic oscillators called Fibonacci oscillators with $SU_r(n)$ -symmetry where $r = q_1/q_2$.

* The work submitted to Int. J. Theo. Phys., (2007) .

- In our construction, the bosonic and fermionic sectors operators commute with each other.

- In our model, the Hamiltonian H_i does not commute with the supercharges Q_i unless $q_1 = q_2 = 1$.

- In the limit $q_1 = q_2 = 1$, the conventional *N=2* SUSY algebra can be recovered (Witten, 1981).

- The limit $q_1 = q_2 = q$ gives an alternative example of the *q*-deformed *N=2* SUSY algebra constructed from the *q*-deformed bosonic and fermionic Newton oscillators (Algin, 2002).

- Recently, the high- and low-temperature thermo-statistical behaviour of a (q_1,q_2)-deformed boson gas with $SU_{q_1/q_2}(2)$-symmetry reveals many interesting results such as the Bose-Einstein condensation in the interval $q_2>q_1>0$ for low temperatures and an interpolation between boson and fermion-like behaviours in some critical values of the deformation parameters (q_1,q_2) for high temperatures.

(A. Algin and B. Deviren, J. Phys. A 38 (2005) 5945;
A. Algin, Phys. Lett. A 292 (2002) 251.)

Some open problems

- In the limit $q_1=q_2=0$, the Fibonacci oscillator algebra gives the Cuntz oscillator algebra as follows:

$$a_i a_k^* = \delta_{ik}, \qquad i,k = 1.2,\ldots,n.$$

(J. Cuntz, Commun. Math. Phys. 57 (1977) 173.)

- A fractional SUSY structure (!) when the deformation parameter $r = q_1/q_2$ is a root of unity.

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