$\mathcal{PT}$-Symmetry and Pseudo-Hermiticity: Mysteries, Facts, and Fiction

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Outline:

- **Pseudo-Hermitian QM & its classical limit**
- **Geometry of State space in Pseudo-Hermitian QM and Faster than Hermitian QM**
- **Pseudo-Hermiticity in classical electrodynamics**
- **Summary and Conclusions**

PHQM & Its Classical Limit:

$\mathcal{H}$: A separable Hilbert Space with defining inner product $\langle \cdot | \cdot \rangle$.

$H : \mathcal{H} \to \mathcal{H}$: A linear (densely defined) operator

Qxn 1: Can H serve as the Hamiltonian operator for a unitary quantum system?

Qxn 2: What are the physical observables?

Qxn 3: How does this quantum theory correspond to Classical Mechanics?

Ansr 1: H can serve as the Hamiltonian operator for a unitary quantum system if and only if there is a positive-definite inner product $\langle\cdot,\cdot\rangle_+$ on $\mathcal{H}$ that renders H self-adjoint, $\langle\psi,H\phi\rangle_+=\langle H\psi,\phi\rangle_+$

Fact: Every positive-definite inner product is given by $\langle\psi,\phi\rangle_+=\langle\psi|\eta_+\phi\rangle$

η_+: **Metric Operator** (positive automorphism = everywhere-defined, bounded, invertible, positive linear operator)

Fact: $\langle\psi,H\phi\rangle_+=\langle H\psi,\phi\rangle_+ \Leftrightarrow H^\dagger=\eta_+H\eta_+^{-1}$

Some Terminology

Def: Let η be a given Hermitian automorphism (a pseudo-metric operator). Then an operator H is called η-Hermitian if $H^\dagger = \eta H \eta^{-1}$.

[W. Pauli, RMP 15, 175 (1943)]

Pauli: Do not consider positive η's, because they lead to a theory that is "equivalent to the usual theory $\cdots$ We get, however, something essentially new if we take into consideration indefinite bilinear forms" (indefinite η.)

Indefinite-Metric QM

H is called pseudo-Hermitian if there is a pseudo-metric operator η satisfying $H^\dagger = \eta H \eta^{-1}$ [A. M., 2002].

Pseudo-Hermitian $\neq$ η-Hermitian, because to determine if H is η-Hermitian, we need η.

H is called quasi-Hermitian if there is a positive automorphism (a metric operator) η_+ satisfying $H^\dagger = \eta_+ H \eta_+^{-1}$ [Scholtz, Geyer, Hahne, 1992.]

So H can serve as the Hamiltonian for a unitary quantum system iff it is quasi-Hermitian.

Use of the methods of indefinite-metric QM in describing $\mathcal{PT}$-symmetric systems [Japaridze, 2002]: $H = p^2 + ix^3$ is $\mathcal{P}$-Hermitian.

$\mathcal{PT}$-symmetry $\neq$ $\mathcal{P}$-Hermiticity, for example $H = p^2+i\{x,p\}$ is $\mathcal{PT}$-symmetric but not $\mathcal{P}$-Hermitian.

If $H^t := \mathcal{T}H^\dagger\mathcal{T} = H$, then $\mathcal{PT}$-symmetry of H is the same as its $\mathcal{P}$-Hermiticity. In this case, if H is diagonalizable and its spectrum is real and discrete, we may apply a construction due to Nevanlinna (1952) to obtain a positive-definite inner product. This is what was rediscovered by Bender, Brody, & Jones in 2002 and called the $\mathcal{CPT}$-inner product.

[CJP 56, 919 (2006); quant-ph/0606173]

Why Pseudo-Hermiticity?

Every diagonalizable $\mathcal{PT}$-symmetric Hamiltonian with a discrete spectrum is pseudo-Hermitian.

Pseudo-Hermiticity = Antilinear Symmetry

Metric Operator for a quasi-Hermitian H:

$$\eta_+ = \sum_n |\phi_n\rangle\langle\phi_n|$$

$$H\psi_n = E_n\psi_n, \qquad H^\dagger\phi_n = E_n\phi_n$$

$$\langle\psi_m|\phi_n\rangle = \delta_{mn}, \qquad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Pseudo-Hermitian QM:

1. Given a quasi-Hermitian Hamiltonian H acting on a reference Hilbert space $\mathcal{H}$ choose a metric operator η_+.

2. Define the Physical Hilbert space $\mathcal{H}_{\rm phys}$ using $\langle\cdot,\cdot\rangle_+ = \langle\cdot|\eta_+\cdot\rangle$.

3. Define the physical observables as self-adjoint operators acting in $\mathcal{H}_{\rm phys}$ (answering Qxn 2).

⋆ $\rho := \sqrt{\eta_+}$ **satisfies** $\langle\psi,\phi\rangle_+ = \langle\rho\,\psi|\rho\,\phi\rangle$. **This means that** $\rho : \mathcal{H}_{\rm phys} \to \mathcal{H}$ **is unitary.**

⋆ $h := \rho\, H\, \rho^{-1}$ **is Hermitian and unitary-equivalent to** H.

⋆ **Every self-adjoint operator** $O : \mathcal{H}_{\rm phys} \to \mathcal{H}_{\rm phys}$ **has the form** $O = \rho^{-1} o \rho$ **for some Hermitian** $o : \mathcal{H} \to \mathcal{H}$.

★ Every pseudo-Hermitian quantum system admits an equivalent Hermitian description using $\mathcal{H}$ as the physical Hilbert space, h as the Hamiltonian, and self-adjoint operators acting in $\mathcal{H}$ as the observables.

★ Pseudo-Hermitian QM is an alternative representation of the conventional QM.

★ The Hermitian Hamiltonian h associated with a local quasi-Hermitian operator H is generally nonlocal.

Quantum-Classical Correspondence

Let Γ be a contour in $\mathbb{C}$, $\mathcal{H} = L^2(\Gamma)$, and

$$H_\Gamma = \frac{\mathfrak{p}^2}{2m} + v(\mathfrak{z}), \qquad \mathfrak{z} \in \Gamma, \qquad \mathfrak{p} = -i\hbar\frac{d}{d\mathfrak{z}}.$$

Map Γ onto $\mathbb{R}$ and use the induced unitary transformation of $L^2(\Gamma)$ onto $L^2(\mathbb{R})$ to describe the system in terms of a (quasi-Hermitian) Hamiltonian $H = H(x,p)$ acting in $L^2(\mathbb{R})$.

JPA 38 (2005) 3213 [quant-ph/0410012]

Choose a metric operator η_+ and determine the corresponding Hermitian Hamiltonian h and the pseudo-Hermitian position X and momentum P operators using $\rho = \sqrt{\eta_+}$:

$$h = \rho H \rho^{-1}, \qquad X = \rho^{-1} x \, \rho, \qquad P = \rho^{-1} p \, \rho$$

Hermitian Rep.: $\mathcal{H}, h, x, p$

Pseudo-Hermitian Rep.: $\mathcal{H}_{\rm Phys}, H, X, P$

h, X, P are generally nonlocal operators.

Classical States: $(x_c, p_c) \in \mathbb{R}^2$

Classical Phase Space: $\mathbb{R}^2$

Classical Hamiltonian:

$$H_c := \lim_{\substack{\hbar \to 0 \\ x \to x_c \\ p \to p_c}} h(x,p) = \lim_{\substack{\hbar \to 0 \\ X \to x_c \\ P \to p_c}} H(X,P)$$

Pseudo-Hermitian Canonical Quantization:

$$x_c \to X, \quad p_c \to P, \quad H_c \to H, \quad \{\cdot,\cdot\}_{\rm PB} \to -i\hbar^{-1}[\cdot,\cdot]$$

Hermitian Canonical Quantization:

$$x_c \to x, \quad p_c \to p, \quad H_c \to h, \quad \{\cdot,\cdot\}_{\rm PB} \to -i\hbar^{-1}[\cdot,\cdot]$$

See JPA 37 (2004) 11645, quant-ph/0408132.

$\mathcal{PT}$-Sym. Cubic Anharmonic Oscillator

[JPA 38 (2005) 6557, quant-ph/0411137]

$$H = \frac{p^2}{2m} + \frac{\mu^2 x^2}{2} + i\,\epsilon\,x^3$$

$$H_c = \frac{p_c^2}{2M(x_c)} + \frac{\mu^2}{2}\,x_c^2 + \frac{3\epsilon^2}{2\mu^2}\,x_c^4 + \mathcal{O}(\epsilon^4),$$

$$M(x_c) := \frac{m}{1 + 3\mu^{-4}\epsilon^2\,x_c^2}.$$

Imaginary Cubic Potential

[JPA. 39, 13495 (2006), quant-ph/0508195]

$$H = \frac{p^2}{2m} + i\,\epsilon\,x^3, \qquad H_c = \frac{p_c^2}{2m} + \frac{3\epsilon^2 x_c^6}{8p_c^2} + \mathcal{O}(\epsilon^4)$$

δ-Function Potential with Complex Coupling

$$H = \frac{p^2}{2m} + \zeta\,\delta(x), \qquad \zeta \in \mathbb{C}, \qquad \Re(\zeta) > 0$$

H is not $\mathcal{PT}$-symmetric.

$$h = \frac{p^2}{2m} + \Re(\zeta)\,\delta(x) + \Im(\zeta)^2\; h_2 + \mathcal{O}(\Im(\zeta)^3)$$

$$h_2\Psi(x) := A_\psi\, e^{-|x|/L} + B_\psi\,\delta(x)$$

$$A_\psi := \frac{m\Psi(0)}{8\hbar^2}, \qquad B_\psi = \frac{m}{8\hbar^2}\int_{-\infty}^{\infty} dx\; e^{-|x|/L}\Psi(x).$$

Length Scale: $L := \dfrac{\hbar^2}{m\,\Re(\zeta)}$

JPA 39 (2006) 13495, quant-ph/0606198

Analytic continuation of CM into $\mathbb{C}$

Consider the following complex extension of the classical Hamiltonian $H_c = \frac{p_c^2}{2m} + v(x_c)$:

$$x_c \longrightarrow \mathfrak{z}, \qquad p_c \longrightarrow \mathfrak{p}, \qquad H_c \longrightarrow \mathfrak{H} = \frac{\mathfrak{p}^2}{2m} + v(\mathfrak{z})$$

where $\mathfrak{z}, \mathfrak{p} \in \mathbb{C}$ and v is analytic, and suppose that Hamilton's equations generate the dynamics:

$$\dot{\mathfrak{z}} = \frac{\partial \mathfrak{H}}{\partial \mathfrak{p}} = \frac{\mathfrak{p}}{m}, \qquad \dot{\mathfrak{p}} = -\frac{\partial \mathfrak{H}}{\partial \mathfrak{z}} = -v'(\mathfrak{z}).$$

[Bender et al, JMP 40, 2201 (1999)]

Such complex dynamical systems have been identified as the classical limit of the quantum systems defined by complex potentials such as $v(x) = ix^3$.

This identification provides a prescription for determining the underlying classical systems for $\mathcal{PT}$-symmetric quantum systems which is fundamentally different from the one used in pseudo-Hermitian QM.

There are two basic difficulties with this prescription.

- The quantum Hamiltonian $H=\frac{p^2}{2m}+v(x)$ defines a system with a one-dimensional configuration space $\mathbb{R}$ and a two-dimensional phase space $\mathbb{R}^2$, whereas the classical Hamiltonian $\mathfrak{H}$ defines a system with a four-dimensional phase space $\mathbb{C}^2=\mathbb{R}^4$.

- The naive canonical quantization,

$$\mathfrak{z}\to x,\quad \mathfrak{p}\to p,\quad \mathfrak{H}\to H,\quad \{\cdot,\cdot\}_{\mathsf{PB}}\to -i\hbar^{-1}[\cdot,\cdot],$$

cannot be consistently performed, because the usual Poisson bracket is incompatible with the dynamical equations.

[Curtright & Mezincescu, quant-ph/0507015]

Most general dynamically compatible symplectic structure:

$w_1 := \Re(\mathfrak{z}),\ w_2 := \Re(\mathfrak{p}),\ w_3 := \Im(\mathfrak{z}),\ w_4 := \Im(\mathfrak{p})$

$$\{\!\{A, B\}\!\} = \sum_{i,j=1}^{4} J_{ij}\,\frac{\partial A}{\partial w_i}\,\frac{\partial B}{\partial w_j},$$

$$J = \frac{1}{2}\begin{pmatrix} 0 & 1+c & -a & -d \\ -(1+c) & 0 & -d & -b \\ a & d & 0 & -1+c \\ d & b & 1-c & 0 \end{pmatrix},$$

$$a, b, c, d \in \mathbb{R}, \qquad c^2 + d^2 - ab \neq 1$$

PLA 357 (2006) 177, quant-ph/0603091

- **Simplest allowed choice:** $a = b = c = d = 0$,

$$J = J_0 := \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

- **For all allowed** a, b, c, d,

$$J \neq J_{\mathrm{st}} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Uniqueness Thm: Up to isomorphisms all symplectic structures on $\mathbb{R}^{2n}$ are equivalent.

There are Darboux coordinates in which J takes the form of J_{st}. For $J = J_0$ these are

$$x_1 := \sqrt{2}\ \Re(\mathfrak{z}) \qquad p_1 := \sqrt{2}\ \Re(\mathfrak{p})$$
$$x_2 := \sqrt{2}\ \Im(\mathfrak{p}) \qquad p_2 := \sqrt{2}\ \Im(\mathfrak{z})$$

[Xavier & de Aguiar, Ann. Phys. 252, 458 (1996)
Kaushal & Singh, Ann. Phys. 288, 253 (2001)]

• In terms of these new coordinates, the dynamics is determined by $\dot{A} = \{A, K_c\}_{PB}$ where the classical Hamiltonian is $K_c := 2\,\Re(\mathfrak{H})$!

• $S_c := \Im(\mathfrak{H})$ generates a symmetry:

- The system is integrable;

- Setting $\Im(\mathfrak{H}) = 0$ means to confine the dynamics to one of the orbits of the corresponding symmetry transformations in the phase space $\mathbb{R}^4$.

$$\mathfrak{H} = \frac{\mathfrak{p}^2}{2m} + v(\mathfrak{z})$$

Example: $\mathfrak{H} = \mathfrak{p}^2 + \mathfrak{z}^2$

$$K_c = p_1^2 + x_1^2 - (p_2^2 + x_2^2)$$

$$S_c = x_1 p_2 + x_2 p_1$$

The plots given by Bender et al [JMP 40, 2201 (1999)] for the orbits in the $\mathfrak{z}$-plane correspond to setting $S_c = 0$ and plotting the projection of the corresponding orbits in the phase space $\mathbb{R}^4$ onto the x_1-p_2 plane.

This is also true for $\mathfrak{H} = p^2 + ix^3$ and other examples considered in the literature.

Example $\mathfrak{H} = \mathfrak{p}^2 + i\mathfrak{z}^3$:

$$K_c = p_1^2 + \frac{p_2^3}{\sqrt{2}} - \frac{3\,x_1^2 p_2}{\sqrt{2}} - x_2^2,$$

$$S_c = x_2 p_1 - \frac{3\,x_1 p_2^2}{2\sqrt{2}} + \frac{x_1^3}{2\sqrt{2}}.$$

Qxn: How is this dynamical system in the phase space $\mathbb{R}^4$ related to the unitary quantum system defined by $H = p^2 + ix^3$ which has phase space $\mathbb{R}^2$?

⋆ One obtains a reduced 2-dimensional phase space if one fixes a gauge on the orbit $S_c = 0$.

⋆ For $v(x) = e^{ix}$, the quantum Hamiltonian is not quasi-Hermitian. It does not define a unitary quantum system and there is no equivalent Hermitian Hamiltonian h and the classical Hamiltonian H_c. Yet one can define a classical dynamics using $\mathfrak{H}$.

⋆ There is an equivalent real description of the continuation on CM to complex plane. One does not get any thing fundamentally new. The situation is very similar to that of pseudo-Hermitian QM.

Time-dependent Quasi-Hermitian Hamiltonians

Suppose $\mathcal{H}_{\rm phys}$ be defined by a time-dependent metric operator $\eta_+(t)$ and $\psi_1(t)$ and $\psi_2(t)$ be arbitrary solutions of the Schrödinger Eqn.

$$i\hbar\frac{d}{dt}\psi(t) = H\psi(t)$$

Then the unitarity of time-evolution, namely $\frac{d}{dt}\langle\psi_1(t),\psi_2(t)\rangle_+ = 0$, is equivalent to

$$H^\dagger = \eta_+ H\eta_+^{-1} - i\hbar^{-1}\eta_+\frac{d}{dt}\eta_+^{-1}.$$

This means that H is not η_+-Hermitian, i.e., as an operator acting in $\mathcal{H}_{\rm phys}$ it is not self-adjoint. So H is not an observable!

The above argument shows that the metric operator defining $\mathcal{H}_{\rm phys}$ must be time-independent. This restricts the choice of H if it is time-dependent, because $\eta_+ = \sum_n |\phi_n\rangle\langle\phi_n|$.

Let $H(t)$ be a time-dependent quasi-Hermitian operator (with a discrete spectrum) and $\mathcal{A}(t)$ be the matrix with entries

$$\mathcal{A}_{mn}(t) := i\langle\phi_m(t)|\frac{d}{dt}|\psi_n(t)\rangle.$$

Then η_+ is time-independent iff $\mathcal{A}(t)^\dagger = \mathcal{A}(t)$. This makes all the adiabatic geometric phase angles real [PLB 650, 208 (2007)].

State Space in Pseudo-Hermitian QM

States are rays in the physical Hilbert spaces $\mathcal{H}_{\rm phys}$. They are points of the **projective Hilbert space** $\mathcal{PH}_{\rm phys}$. Each state may be represented by a nonzero vector $\psi \in \mathcal{H}_{\rm phys}$ but this is not a unique representation. A unique representation is provided by the **orthogonal projection operator** Λ onto the ray. This **is an observable satisfying $\Lambda^2 = \Lambda$**. It is give by

$$\Lambda := \frac{|\psi\rangle\langle\psi|\eta_+}{\langle\psi,\psi\rangle_+} = \frac{|\psi\rangle\langle\psi|\eta_+}{\langle\psi|\eta_+\psi\rangle}$$

$\langle\psi|$ stands for the functional $f_\psi : \mathcal{H}_{\rm phys} \to \mathbb{C}$ defined by $f_\psi(\phi) := \langle\psi|\phi\rangle$.

The projection operators $\Lambda_\psi := \frac{|\psi\rangle\langle\psi|\eta_+}{\langle\psi,\psi\rangle_+}$ also satisfy the condition that orthogonal state vectors define orthogonal states, i.e.,

$$\langle\psi,\phi\rangle_+ = 0 \;\Leftrightarrow\; \Lambda_\psi\Lambda_\phi = \Lambda_\phi\Lambda_\psi = 0.$$

This is a requirement of the projection (measurement) axiom of QM. It induces a particular geometric structure on $\mathcal{P}\mathcal{H}_{\rm phys}$.

For $\mathcal{H} = \mathbb{C}^2$, we have

$$\eta_+ = \begin{pmatrix} a & \beta \\ \beta^* & c \end{pmatrix},$$

$$a, c \in \mathbb{R}^+,\ \ \beta \in \mathbb{C},\ \ ac > |\beta|^2$$

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow \Lambda_1 = \begin{pmatrix} 1 & \frac{\beta}{a} \\ 0 & 0 \end{pmatrix}$$

$$\psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow \Lambda_2 = \begin{pmatrix} 0 & 0 \\ \frac{\beta^*}{c} & 1 \end{pmatrix}$$

To determine the geometry of $\mathcal{PH}_{\text{phys}}$ we first define an inner product on the space of trace-class operators acting in $\mathcal{H}_{\text{phys}}$:

$$(A, B) := \text{tr}(A^\sharp B), \qquad A^\sharp := \eta_+^{-1} A^\dagger \eta_+.$$

This ensures: $\langle\psi, \phi\rangle_+ = 0 \;\Leftrightarrow\; (\Lambda_\psi, \Lambda_\phi) = 0.$

$(\cdot, \cdot)$ is the unique inner product such that given an orthonormal basis $\{\psi_n\}$ of $\mathcal{H}_{\text{phys}}$ the corresponding states are orthonormal:

$$\langle\psi_m, \psi_n\rangle_+ = \delta_{mn} \;\Leftrightarrow\; (\Lambda_{\psi_m}, \Lambda_{\psi_n}) = \delta_{mn}.$$

$$\Lambda_\psi := \frac{|\psi\rangle\langle\psi|\eta_+}{\langle\psi,\psi\rangle_+}$$

The geometry of $\mathcal{PH}_{\text{phys}}$ is given by

$$ds^2 := (d\Lambda_\psi, d\Lambda_\psi) = \text{tr}(d\Lambda_\psi^\sharp d\Lambda_\psi)$$
$$= \frac{2[\langle\psi,\psi\rangle_+ \langle d\psi, d\psi\rangle_+ - |\langle\psi, d\psi\rangle_+|^2]}{|\langle\psi,\psi\rangle_+|^2}.$$

If $\mathcal{H} = \mathbb{C}^N$ with Euclidean inner product, $\psi = (z^1, \cdots, z^N)^t$, $\eta_+ = (\eta_{ij})$, and

$$ds^2 = g_{ij^*} dz^i dz^{j^*},$$
$$g_{ij^*} := \frac{2\sum_{p,q=1}^N \left(\eta_{pq}\eta_{ji} - \eta_{pi}\eta_{jq}\right) z_p^* z_q}{\left(\sum_{m,n=1}^N \eta_{mn} z_m^* z_n\right)^2}.$$

For $\eta_+ = 1$, i.e., the ordinary QM this gives the Fubini-Study metric on $\mathcal{PH} = \mathbb{C}P^{N-1}$.

To compare the geometry of $\mathcal{PH}_{\text{phys}}$ with $\mathcal{PH}$ we use the unitary map $\rho : \mathcal{H}_{\text{phys}} \to \mathcal{H}$ to define $f : \mathcal{PH}_{\text{phys}} \to \mathcal{PH}$ according to

$$f(\Lambda) := \rho\Lambda\rho^{-1}.$$

This function turns out to be an isometry. This means that $\mathcal{PH}_{\text{phys}}$ and $\mathcal{PH}$ have identical geometries. In particular the geodesic distance between Λ_1 and Λ_2 in $\in \mathcal{PH}_{\text{phys}}$ will be identical with the geodesic distance between $f(\Lambda_1)$ and $f(\Lambda_2)$ in $\mathcal{PH}$.

For $\mathcal{H} = \mathbb{C}^2$ both $\mathcal{PH}_{\text{phys}}$ and $\mathcal{PH}$ are (round) unit two-dimensional spheres.

To setup the variational problem of finding a quasi-Hermitian Hamiltonian that generates fastest possible unitary time-evolution from an initial state vector ψ_I to a final state vector ψ_F involves fixing the boundary conditions:

$$\Lambda(t=0)=\Lambda_{\psi_I}, \qquad \Lambda(t=\tau)=\Lambda_{\psi_F}$$

But Λ_{ψ_I} and Λ_{ψ_F} know about the choice of the metric operator. More importantly the distance between Λ_{ψ_I} and Λ_{ψ_F} depends on η_+.

Different H will require different η_+ and lead to different distances between Λ_{ψ_I} and Λ_{ψ_F}.

The Hermitian mirror image of dynamics:

$$\underbrace{\psi_I \xrightarrow{H} \psi_F}_{\text{in time } \tau} \quad \Leftrightarrow \quad \underbrace{\rho\,\psi_I \xrightarrow{h} \rho\,\psi_F}_{\text{in time } \tau}$$

$$\underbrace{\Lambda_{\psi_I} \xrightarrow{H} \Lambda_{\psi_F}}_{\text{in time } \tau} \Leftrightarrow \underbrace{f(\Lambda_{\psi_I}) \xrightarrow{h} f(\Lambda_{\psi_F})}_{\text{in time } \tau}$$

The speed of the evolution is the distance between the initial and final states divides by τ. distance$(\Lambda_{\psi_I}, \Lambda_{\psi_F})$=distance$(f(\Lambda_{\psi_I}), f(\Lambda_{\psi_F})) \Rightarrow$ the optimal speed that is achieved by H is also achieved by h; The upper bound on the speed of unitary evolutions is the same for both Hermitian and non-Hermitian Hamiltonians. You cannot go faster than Hermitian QM unless you violate unitarity.

The impossibility of faster than Hermitian QM cannot be avoided by arguments that involve use of Hermitian and non-Hermitian Hamiltonians, i.e., "switching Hilbert spaces," because this requires time-dependent metric operators that are forbidden by unitarity. You can only achieve faster evolutions in the cost of sacrificing unitarity.

$$t_1 \xrightarrow{h_1} t_2 \xrightarrow{H} t_3 \xrightarrow{h_2} t_4$$

h_1, h_2: Hermitian
H: Non-Hemritian

[arXiv:0706.3844]

Pseudo-Hermiticity in Electrodynamics

Source-Free Maxwell's Eqs.:

$$\vec{\nabla}\cdot\vec{D}=0, \qquad \vec{\nabla}\cdot\vec{B}=0,$$

$$(\star) \qquad \dot{\vec{B}}+\mathfrak{D}\vec{E}=0, \qquad \dot{\vec{D}}-\mathfrak{D}\vec{H}=0,$$

$$\vec{D}:=\overleftrightarrow{\varepsilon}\,\vec{E}, \qquad \vec{H}:=\overleftrightarrow{\mu}'\vec{B}, \qquad \mathfrak{D}\vec{F}:=\vec{\nabla}\times\vec{F}$$

$$(\star) \;\Rightarrow\; \ddot{\vec{E}}+\Omega^2\vec{E}=0,$$

$$\Omega^2:=\overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}'\mathfrak{D}.$$

$$\ddot{\vec{E}} + \Omega^2 \vec{E} = 0$$

$$\Downarrow$$

$$\vec{E}(\vec{x},t) = \cos(\Omega t)\vec{E}_0(\vec{x}) + \Omega^{-1}\sin(\Omega t)\dot{\vec{E}}_0(\vec{x})$$

$$\cos(\Omega t) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,(t^2\Omega^2)^n,$$

$$\Omega^{-1}\sin(\Omega t) := t\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,(t^2\Omega^2)^n.$$

$$\dot{\vec{E}}_0(\vec{x}) := \dot{\vec{E}}(\vec{x},0) = \overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}'\vec{B}_0(\vec{x}),$$

$$\Omega^2 := \overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}'\mathfrak{D}$$

- $\mathfrak{D}:\mathcal{H}\to\mathcal{H}$ is Hermitian, where

$$\mathcal{H}:=\{\vec{F}:\mathbb{R}^3\to\mathbb{C}^3|\prec\vec{F},\vec{F}\succ<\infty\}$$
$$\prec\vec{F},\vec{G}\succ:=\int_{\mathbb{R}^3}\vec{F}(\vec{x})^*\cdot\vec{G}(\vec{x})\ dx^3$$

- If $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}'$ are Hermitian and $\overleftrightarrow{\varepsilon}$ is invertible, $\Omega^2:\mathcal{H}\to\mathcal{H}$ is pseudo-Hermitian:

$$(\Omega^2)^\dagger=\overleftrightarrow{\varepsilon}\,\Omega^2\,\overleftrightarrow{\varepsilon}^{-1}$$

- If $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}'$ are positive (lossless material), Ω^2 is quasi-Hermitian. $\Rightarrow$ It is unitary-equivalent to the positive (Hermitian) operator:

$$h:=\overleftrightarrow{\varepsilon}^{\frac{1}{2}}\Omega^2\overleftrightarrow{\varepsilon}^{-\frac{1}{2}}=\overleftrightarrow{\varepsilon}^{-\frac{1}{2}}\mathfrak{D}\,\overleftrightarrow{\mu}'\mathfrak{D}\,\overleftrightarrow{\varepsilon}^{-\frac{1}{2}}$$

We can compute any (even analytic) function $\mathcal{F}$ of Ω using the spectral resolution of h:

$$\mathcal{F}(\Omega) = \mathcal{F}(\overleftrightarrow{\varepsilon}^{-\frac{1}{2}} h^{\frac{1}{2}} \overleftrightarrow{\varepsilon}^{\frac{1}{2}}) = \overleftrightarrow{\varepsilon}^{-\frac{1}{2}} \mathcal{F}(h^{\frac{1}{2}}) \overleftrightarrow{\varepsilon}^{\frac{1}{2}}$$

This allows for expressing $\cos(\Omega t)$ and $\Omega^{-1} \sin(\Omega t)$ as certain integral operators and computing their kernels (propagators). Recall that

$$\vec{E}(\vec{x}, t) = \cos(\Omega t) \vec{E}_0(\vec{x}) + \Omega^{-1} \sin(\Omega t) \dot{\vec{E}}_0(\vec{x})$$

Planar waves propagating along the z-axis:

$$\vec{E}_0 = \mathcal{E}(z)e^{-ik_0 z}\ \hat{i}, \qquad \vec{B}_0 = \mathcal{B}(z)e^{-ik_0 z}\ \hat{j}$$

Isotropic media with:

$$\overleftrightarrow{\varepsilon}(\vec{x}) = \varepsilon(z)\overleftrightarrow{1}, \qquad \overleftrightarrow{\mu}'(\vec{x}) = \mu(z)^{-1}\overleftrightarrow{1}$$

h is a position-dependent-mass Hamiltonian:

$$h = \varepsilon(z)^{-\frac{1}{2}} p\, \mu(z)^{-1} p\, \varepsilon(z)^{-\frac{1}{2}}$$

Suppose $\varepsilon, \mu \to$ const. as $|z| \to \infty$.

WKB Approximation: $h\,\psi_\omega = \omega^2\,\psi_\omega$

$$\psi_\omega(z) \approx \frac{e^{i\omega u(z)}}{\sqrt{2\pi\, v(z)}}, \qquad \omega \in \mathbb{R}$$

$$u(z) := \int_0^z \frac{d\mathfrak{z}}{v(\mathfrak{z})}, \qquad v(z) := [\varepsilon(z)\mu(z)]^{-\frac{1}{2}}$$

$$\cos(h^{\frac{1}{2}}t) = \int_{-\infty}^{\infty} d\omega\ \cos(\omega t)\ |\psi_\omega\rangle\langle\psi_\omega|,$$

$$h^{-\frac{1}{2}}\sin(h^{\frac{1}{2}}t) = \int_{-\infty}^{\infty} d\omega\ \frac{\sin(\omega t)}{\omega}\ |\psi_\omega\rangle\langle\psi_\omega|.$$

$$\vec{E}(z,t) = \frac{1}{2}\left[\frac{\mu(z)}{\varepsilon(z)}\right]^{\frac{1}{4}}\left\{\left[\frac{\varepsilon(w_-(z,t))}{\mu(w_-(z,t))}\right]^{\frac{1}{4}}\vec{E}_0(w_-(z,t)) + \left[\frac{\varepsilon(w_+(z,t))}{\mu(w_+(z,t))}\right]^{\frac{1}{4}}\vec{E}_0(w_+(z,t)) + \int_{w_-(z,t)}^{w_+(z,t)} dw\ \mu(w)^{\frac{1}{4}}\varepsilon(w)^{\frac{3}{4}}\dot{\vec{E}}_0(w)\right\}.$$

$$w_\pm(z,t) := u^{-1}(u(z) \pm t)$$

$$u(z) := \int_0^z \frac{d\mathfrak{z}}{v(\mathfrak{z})}, \qquad v(z) := [\varepsilon(z)\mu(z)]^{-\frac{1}{2}}$$

A. M. & F. Loran: physics/0703080

We have used this formalism to study the scattering of planar EM waves off a localized inhomogeneity given by

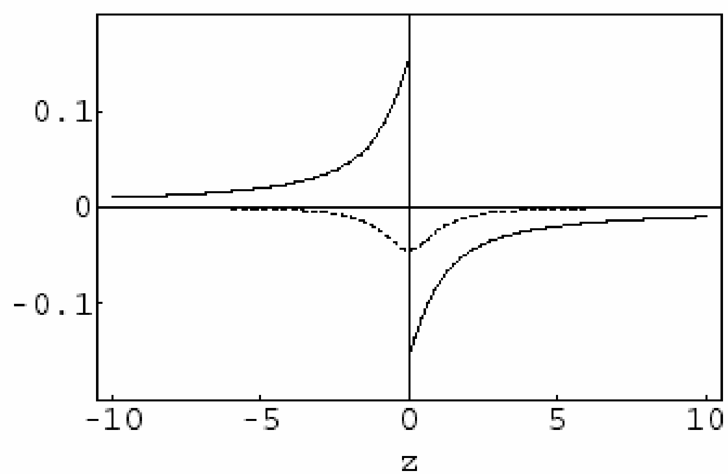
$$\varepsilon(z) = \varepsilon_0 \left(1 + \frac{a}{1 + z^2/\gamma^2}\right), \qquad \mu = \mu_0.$$

$$\vec{E}_{scatt} = \vec{E}\Big|_{a=0} \ \rho(z)\, e^{-ik_0 r(z)}$$

$r(z)$ (full curve)

$\rho(z)$ (dotted curve)

$a = 0.2$, $\gamma = 1$



Summary & Conclusions

- Pseudo-Hermitian QM (in particular $\mathcal{PT}$-symmetric QM) is an alternative representation of QM. It allows for treating a certain class of nonlocal Hermitian Hamiltonians in terms of non-Hermitian but local Hamiltonians.

- The essential property for a non-Hermitian Hamiltonian to generate a unitary evolution is its quasi-Hermiticity. $\mathcal{PT}$-symmetry is neither necessary nor sufficient. A good example is the delta-function potential with a complex coupling which is manifestly non-$\mathcal{PT}$-symmetric.

- The naive extension of CM to complex domain yields integrable systems in a 4-dim. phase space. These may be reduced to a system with a 2-dim. phase space but there seems to be no consistent quantization scheme that would map such a system to the analogous quantum systems.

- The study of the classical trajectories in the complex x-plane for $H = p^2 + v(x)$ with v a complex-valued potential corresponds to looking at the projection of the phase space orbits of an integrable real Hamiltonian with phase space $\mathbb{R}^4 = \{(x_1, x_2, p_1, p_2)\}$ onto the x_1-p_2 plane. There seems to be no point in doing this, as there is a well-known theory of integrable systems.

- The equivalence of Hermitian and pseudo-Hermitian representations of QM manifests itself in terms of the existence of an isometry between the corresponding state spaces. This is to be expected because the geometry of the state space is linked with various physical quantities.

- The upper bound on the speed of unitary evolutions is a universal quantity independent of whether one uses Hermitian or pseudo-Hermitian representation. Because these two representations are equivalent (both mathematically and physically) one cannot differentiate between them using physical quantities.

- Pseudo-Hermiticity arise in a variety of subjects including classical electrodynamics. The methods of pseudo-Hermitian QM give rise to a closed form expression for the semi-classical solution of the initial-value problem for source-free Maxwell's equations in arbitrary stationary linear dielectric media.

Thank You for Your Attention

"Important scientific discoveries go through three phases: first they are completely ignored, then they are violently attacked, and finally they are brushed aside as well-known."

Konrad Lorenz
(Animal Behaviorist)