PT-symmetric potentials with position-dependent mass

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Effective-mass (EM) eigenvalue equation (in one dimension)

\[ H_{\text{EM}}(x)\psi(x) \equiv [T_{\text{EM}}(x) + V_{\text{EM}}(x)]\psi(x) = \varepsilon\psi(x) \]

- \( V_{\text{EM}}(x) \rightarrow \) Local potential strength
- \( T_{\text{EM}}(x) \rightarrow \) EM Kinetic-energy operator (KO)

- \( p(x) = -i\hbar \frac{d}{dx} \) and \( m(x) \) do not commute

\( \implies \) An ambiguity arises in the representation of \( T_{\text{EM}}(x) \)
Some forms of $T_{EM}(x)$

1. $p \frac{1}{2m(x)} p$

2. $\frac{1}{4m(x)} p^2 + p^2 \frac{1}{4m(x)}$

3. $\frac{1}{2 \sqrt{m(x)}} p^2 \frac{1}{\sqrt{m(x)}}$

Contexts & Refs.

Space-Charge effects on electron tunneling in semimetal & semiconductor junctions [Phys. Rev. 152 (1966) 683]


Interface connection rules for wave functions at an abrupt heterojunction between two different semiconductors [Phys. Rev. B27 (1983) 3579]
General two-parametric representation of KO

\[ T_{EM}(x) = \frac{1}{4} \left( m^{\xi} p m^{\eta} p m^{\zeta} + m^{\zeta} p m^{\eta} p m^{\xi} \right) \]
\[ \xi + \eta + \zeta = -1 \]

\[
= \left\{ \begin{array}{ll}
\frac{1}{2m(x)} \frac{1}{p} p \\
\frac{1}{4m(x)} \frac{1}{p^2} + \frac{1}{4m(x)} \frac{1}{p} \frac{1}{p} \\
\frac{1}{2\sqrt{m(x)}} \frac{1}{p^2} \frac{1}{\sqrt{m(x)}} \\
\end{array} \right. 
\]
\[
\xi = \zeta = 0, \eta = -1 \\
\xi = -1, \eta = \zeta = 0 \\
\xi = \zeta = -\frac{1}{2}, \eta = 0
\]
Final form of EM Schroedinger equation

In the atomic units defined by $\hbar^2 = 2$

$$H_{EM}(x)\psi(x) \equiv \left[ -\partial \left( \frac{1}{m} \partial \right) + \tilde{V}_{EM}(x) \right] \psi(x) = \epsilon \psi(x)$$

$$\tilde{V}_{EM}(x) = V_{EM}(x) + \rho(m)$$

$$\rho(m) = \frac{1 + \eta m''}{2} \frac{m'}{m^2} - \left[ 1 + \eta + \xi (\xi + \eta + 1) \right] \frac{m'^2}{m^3}$$
Supersymmetric approach

\[ H_s = \{ Q, Q^\dagger \}, \quad Q^2 = Q^{\dagger 2} = 0, \]
\[ [Q, H_s] = [Q^\dagger, H_s] = 0 \]

\[ Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad H_s = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \]

\[ H_+ = A^\dagger A = -\partial \left( \frac{1}{m} \partial \right) + V_+(x), \]
\[ H_- = AA^\dagger = -\partial \left( \frac{1}{m} \partial \right) + V_-(x), \]
Intertwining relations

\[ AH_+ = H_- A, \quad H_+ A^\dagger = A^\dagger H_. \]

\[ \sqrt{E_{n+1}^+} \psi_n^-(x) = A \psi_{n+1}^+(x), \quad E_n^- = E_{n+1}^+, \]
\[ \sqrt{E_n^-} \psi_{n+1}^+(x) = A^\dagger \psi_n^-(x), \quad E_0^+ = 0. \]

\[ V_+(x) = \tilde{V}_{\text{EM}}(x) - \epsilon, \quad \epsilon \leq E_0^+ = 0 \]
Representation of ladder operators

First order representation is not unique

1. \[ A = \frac{1}{\sqrt{m(x)}} \partial + W(x), \]

2. \[ A = \frac{m^\alpha \partial m^\beta + bm^\beta \partial m^\alpha}{b + 1} + W(x) \]
   \[ (\alpha + \beta = -1/2) \]

3. \[ A = \frac{1}{m^{1/4}} \partial \frac{1}{m^{1/4}} + W(x) \]

Final form of EM partner potentials

\[ V_+(x) = W^2 - \left[ \frac{W(x)}{\sqrt{m(x)}} \right]' \]

\[ A = \frac{1}{\sqrt{m(x)}} \partial + W(x) \]

\[ V_-(x) = V_+(x) + 2 \frac{W'(x)}{\sqrt{m(x)}} - \frac{1}{\sqrt{m(x)}} \left[ \frac{1}{\sqrt{m(x)}} \right]'' \]

Zero-mode states

\[ \psi^+_0(x) \propto \exp \left[ - \int^x \sqrt{m(\tau)} W(\tau) \, d\tau \right] \]

\[ \psi^-_0(x) \propto \sqrt{m(x)} \exp \left[ \int^x \sqrt{m(\tau)} W(\tau) \, d\tau \right] \]
• We now allow the superpotential to be complex
• Consequently, all the operation of hermitian-conjugation will be replaced by transposition

\[ W(x) = f(x) + ig(x) \quad f, g : \mathbb{R} \rightarrow \mathbb{R} \]

\[
V_+(x) = \left[ f^2 - g^2 - \left( \frac{f}{\sqrt{m}} \right)' \right] + i \left[ 2fg - \left( \frac{g}{\sqrt{m}} \right)' \right]
\]

\[
V_-(x) = \left[ f^2 - g^2 + \frac{f'}{\sqrt{m}} - f \left( \frac{1}{\sqrt{m}} \right)' - \frac{1}{\sqrt{m}} \left( \frac{1}{\sqrt{m}} \right)'' \right] + i \left[ 2fg - \left( \frac{g}{\sqrt{m}} \right)' + 2\frac{g'}{\sqrt{m}} \right]
\]
• Our strategy is to force one partner potential to be strictly real, while other will remain complex.

Suppose now $V_+(x)$ is real

$$\left( \frac{g}{\sqrt{m}} \right)' - 2fg = 0$$

Solutions:

$$f \neq g: \quad g(x) = \lambda \sqrt{m(x)} \exp \left[ 2 \int_{x}^{x} f(\tau) \sqrt{m(\tau)} \, d\tau \right]$$

$$f = g: \quad g(x) = -\frac{\sqrt{m(x)}}{\lambda + 2 \int_{x}^{x} m(\tau) \, d\tau}$$
• Thus if $V_+(x)$ is known, then we have a new complex potential with real spectra.

• $V_+(x) = f^2 - g^2 - (f/\sqrt{m})'$

• $V_-(x) = [V_+(x) + 2(f'/\sqrt{m}) - (1/\sqrt{m})''/\sqrt{m}] + i[2g'/\sqrt{m}]$

• To solve the spectral problem for $H_+(x)$, we may use well-known method of point-canonical transformation
Mapping to constant-mass problem

\[ H_+(x) \psi^+(x) \equiv \left[ -\partial \left( \frac{1}{m} \partial \right) + V_+(x) - E^+ \right] \psi^+(x) = 0 \]

\[ z = \int^x \sqrt{m(\tau)} \, d\tau, \quad \chi^+(z) = [m(x)]^{-1/4} \psi(x) \bigg|_{x=z} \]

\[ \left[ -\frac{d^2}{dz^2} + \tilde{V}_+(z) - E^+ \right] \chi^+(z) = 0 \]

\[ \tilde{V}_+(z) = V^+(x) - \frac{m''(x)}{4m^2(x)} + \frac{7}{16} \frac{m'^2(x)}{m^3(x)} \bigg|_{x=z} \]
A simple example for $f = 0$

- In this case $W(x) = i\lambda \sqrt{m(x)}$, \quad $V_+(x) = -\lambda^2 m(x)$

- Above relation between potential and mass may be exploited to get a solvable problem

For example, choose the following mass function and the superpotential

$$m(x) = \frac{1}{\alpha^2(1 + x^2)}, \quad \alpha > 0$$

$$W(x) = i \frac{\mu + \alpha/2}{\sqrt{1 + x^2}}, \quad \mu > 0$$
Corresponding constant-mass equation

\[
\left[ -\frac{d^2}{dz^2} + \tilde{V}_+(z) - \tilde{E}_n^+ \right] \chi_n^+(z) = 0, \quad z = \sinh^{-1} x
\]

\[
\tilde{V}_+(z) = \frac{\alpha^2}{4} - \mu (\mu + \alpha) \text{sech}^2 \alpha z,
\]

\[
\tilde{E}_n^+ (z) = \frac{\alpha^2}{4} - (\mu - n\alpha)^2, \quad n \leq [\mu/\alpha]
\]

\[
\chi_n^+(z) = \text{sech}^\mu (\alpha z) P_n^{(-\frac{\mu}{\alpha} - \frac{1}{2}, -\frac{\mu}{\alpha} - \frac{1}{2})}(i \sinh \alpha z)
\]

- Note that the ground state $\chi_0^+(z)$ does not correspond the zero energy state of EM Hamiltonian $H_+$.

- This implies that both partner Hamiltonians are strictly isospectral, though the SUSY is unbroken due to the existence of both vacuum. This type of peculiarity is known for constant-mass $\mathcal{PT}$-symmetric problems.
Wave functions and spectra of $H_{\pm}(x)$

\[ V_+(x) = -\frac{(\mu + \alpha/2)^2}{1 + x^2}, \quad m(x) = \frac{1}{\alpha^2} \frac{1}{1 + x^2} \]

\[ \psi_0^+(x) \propto \exp \left[ -i \left( \frac{\mu}{\alpha} + \frac{1}{2} \right) \tan^{-1} x \right], \quad E_0^+ = 0, \quad E_{n+1}^+ = \frac{\alpha^2}{4} - (\mu - n\alpha)^2 \]

\[ \psi_{n+1}^+(x) \propto (1 + x^2)^{-\frac{(2\mu + \alpha)}{4\alpha}} P_n^{(-\frac{\mu}{\alpha} - \frac{1}{2}, -\frac{\mu}{\alpha} - \frac{1}{2})} (ix) \]

Target potential:

\[ V_-(x) = -\frac{\mu(\mu + \alpha) + 5\alpha^2/4}{1 + x^2} - i\alpha \frac{(2\mu + \alpha)x}{1 + x^2} \]

\[ \psi_0^-(x) \propto (1 + x^2)^{-1/2} \exp \left[ i \left( \frac{\mu}{\alpha} + \frac{1}{2} \right) \tan^{-1} x \right], \quad \psi_{n+1}^-(x) \propto A \psi_{n+1}^+(x), \]

\[ E_n^- = E_n^+, \quad A = \alpha \sqrt{1 + x^2} \frac{d}{dx} + i \frac{\mu + \alpha/2}{\sqrt{1 + x^2}} \]

Property:

\[ V_{\pm}^*(-x) = V_{\pm}(x), \quad (\psi_n^\pm)^*(-x) = \psi_n^\pm(x) \]
We have thus obtained following new class of $PT$-symmetric potential and mass with real spectra

\[
V(x) = -\frac{\eta_1}{1 + x^2} - i\eta_2 \frac{x}{1 + x^2}, \quad m(x) = \frac{1}{\alpha^2(1 + x^2)}
\]

- Is there any restriction between the parameters for the existence of real spectra?
- To answer this question, it is convenient to look into the corresponding constant-mass $PT$-symmetric potential

\[
\tilde{V}(z) = -\tilde{\eta}_1 \text{sech}^2 \alpha z - i\tilde{\eta}_2 \text{sech} \alpha z \tanh \alpha z, \quad \tilde{\eta}_1 = \eta_1 - \frac{\alpha^2}{4}, \tilde{\eta}_2 = \eta_2
\]

- Note that for this potential, it is well-known that [Phys. Lett. A 282 (2001) 343] the coupling constants must satisfy following inequality to keep $PT$-symmetry unbroken
\[ |\tilde{\eta}_2| \leq \tilde{\eta}_1 + \frac{\alpha^2}{4}, \quad \tilde{\eta}_1 > 0 \]

- Hence, for our $\mathcal{P}\mathcal{T}$-symmetric EM Hamiltonian, corresponding condition reads

\[ |\eta_2| \leq \eta_1, \quad \eta_1 > 0 \]

Note that for the present example, this restriction is trivially true.

Thank you for your kind attention