PT-symmetric potentials with position-dependent mass

Asish Ganguly

University of Valladolid, Spain
University of Calcutta, India

Effective-mass (EM) eigenvalue equation
(in one dimension)

$$\boxed{H_{\mathrm{EM}}(x)\psi(x) \equiv [T_{\mathrm{EM}}(x) + V_{\mathrm{EM}}(x)]\psi(x) = \epsilon\psi(x)}$$

$V_{\mathrm{EM}}(x) \longrightarrow$ Local potential strength

$T_{\mathrm{EM}}(x) \longrightarrow$ EM Kinetic-energy operator (KO)

- $p(x) = -i\hbar\frac{d}{dx}$ and $m(x)$ do not commute

$\Rrightarrow$ An ambiguity arises in the representation of $T_{\mathrm{EM}}(x)$

Some forms of $\mathrm{T}_{\mathrm{EM}}(\mathrm{x})$

Contexts & Refs.

1. $p\frac{1}{2m(x)}p$

Space-Charge effects on electron tunneling in semimetal & semiconductor junctions [Phys. Rev. 152 (1966) 683]

2. $\frac{1}{4m(x)}p^2 + p^2\frac{1}{4m(x)}$

Transport in graded mixed semiconductor [Phys. Rev. 177 (1969) 1179]

3. $\frac{1}{2\sqrt{m(x)}}p^2\frac{1}{\sqrt{m(x)}}$

Interface connection rules for wave functions at an abrupt heterojunction between two different semiconductors [Phys. Rev. B27 (1983) 3579]

General two-parametric representation of KO

[Phys. Rev. B27(1983) 7547]

$$T_{\mathrm{EM}}(x) = \frac{1}{4}\left(m^{\xi} p\, m^{\eta} p\, m^{\zeta} + m^{\zeta} p\, m^{\eta} p\, m^{\xi}\right)$$

$$\xi + \eta + \zeta = -1$$

$$= \begin{cases} p\dfrac{1}{2m(x)}p & \xi = \zeta = 0, \eta = -1 \\ \dfrac{1}{4m(x)}p^2 + p^2\dfrac{1}{4m(x)} & \xi = -1, \eta = \zeta = 0 \\ \dfrac{1}{2\sqrt{m(x)}}p^2\dfrac{1}{\sqrt{m(x)}} & \xi = \zeta = -\frac{1}{2}, \eta = 0 \end{cases}$$

Final form of EM Schroedinger equation

In the atomic units defined by $\hbar^2 = 2$

$$H_{\mathrm{EM}}(x)\psi(x) \equiv [-\partial(\frac{1}{m}\partial) + \widetilde{V}_{\mathrm{EM}}(x)]\psi(x) = \epsilon\psi(x)$$

$$\widetilde{V}_{\mathrm{EM}}(x) = V_{\mathrm{EM}}(x) + \rho(m)$$

$$\rho(m) = \frac{1+\eta}{2}\frac{m''}{m^2} - [1+\eta+\xi(\xi+\eta+1)]\,\frac{m'^2}{m^3}$$

Supersymmetric approach

$$H_s = \{Q, Q^\dagger\}, \qquad Q^2 = Q^{\dagger^2} = 0,$$

$$[Q, H_s] = [Q^\dagger, H_s] = 0$$

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad H_s = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

$$\Rightarrow \quad H_+ = \mathrm{A}^\dagger \mathrm{A} = -\partial\left(\frac{1}{m}\partial\right) + V_+(x)\,,$$

$$H_- = \mathrm{A}\mathrm{A}^\dagger = -\partial\left(\frac{1}{m}\partial\right) + V_-(x)\,,$$

Intertwining relations

$$\boxed{A H_+ = H_- A, \qquad H_+ A^\dagger = A^\dagger H_- .}$$

$$\sqrt{E_{n+1}^+}\,\psi_n^-(x) = A\psi_{n+1}^+(x), \quad E_n^- = E_{n+1}^+,$$

$$\sqrt{E_n^-}\,\psi_{n+1}^+(x) = A^\dagger \psi_n^-(x)\,, \quad E_0^+ = 0\,.$$

$$V_+(x) = \widetilde{V}_{\mathrm{EM}}(x) - \epsilon, \qquad \epsilon \leqslant E_0^+ = 0$$

Representation of ladder operators

First order representation is not unique

1. $$A = \frac{1}{\sqrt{m(x)}}\partial + W(x),$$

- Phys. Rev. A60 (1999) 4318

2. $$\mathrm{A} = \frac{m^{\alpha}\partial m^{\beta} + b m^{\beta}\partial m^{\alpha}}{b+1} + W(x)$$

$$(\alpha + \beta = -1/2)$$

- Europhys. Lett. 62 (2003) 8

3. $$\mathrm{A} = \frac{1}{m^{1/4}}\partial\frac{1}{m^{1/4}} + W(x)$$

- J. Phys. A 38 (2005) 2929

Final form of EM partner potentials

$$V_+(x) = W^2 - \left[\frac{W(x)}{\sqrt{m(x)}}\right]'$$

$$\boxed{A = \frac{1}{\sqrt{m(x)}}\partial + W(x)}$$

$$V_-(x) = V_+(x) + 2\frac{W'(x)}{\sqrt{m(x)}} - \frac{1}{\sqrt{m(x)}}\left[\frac{1}{\sqrt{m(x)}}\right]''$$

Zero-mode states

$$\psi_0^+(x) \propto \exp\left[-\int^x \sqrt{m(\tau)}W(\tau)\,\mathrm{d}\tau\right]$$

$$\psi_0^-(x) \propto \sqrt{m(x)}\exp\left[\int^x \sqrt{m(\tau)}W(\tau)\,\mathrm{d}\tau\right]$$

- We now allow the superpotential to be complex
- Consequently, all the operation of hermitian-conjugation will be replaced by transposition

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$$W(x) = f(x) + ig(x) \quad f, g : \mathbb{R} \rightarrow \mathbb{R}$$

$$V_+(x) = \left[f^2 - g^2 - \left(\frac{f}{\sqrt{m}} \right)' \right] + i \left[2fg - \left(\frac{g}{\sqrt{m}} \right)' \right]$$

$$V_-(x) = \left[f^2 - g^2 + \frac{f'}{\sqrt{m}} - f \left(\frac{1}{\sqrt{m}} \right)' - \frac{1}{\sqrt{m}} \left(\frac{1}{\sqrt{m}} \right)'' \right] + i \left[2fg - \left(\frac{g}{\sqrt{m}} \right)' + 2 \frac{g'}{\sqrt{m}} \right]$$

- Our strategy is to force one partner potential to be strictly real, while other will remain complex.

Suppose now $V_+(x)$ is real

$$\Rightarrow \left(\frac{g}{\sqrt{m}}\right)' - 2fg = 0$$

Solutions :

$$f \neq g: \;\; g(x) = \lambda \sqrt{m(x)} \exp\left[2\int^x f(\tau)\sqrt{m(\tau)}\, d\tau\right]$$

$$f = g: \;\; g(x) = -\frac{\sqrt{m(x)}}{\lambda + 2\int^x m(\tau)\, d\tau}$$

- Thus if $V_+(x)$ is known, then we have a new complex potential with real spectra.

- $V_+(x) = f^2 - g^2 - (f/\sqrt{m})'$

- $$V_-(x) = [V_+(x) + 2(f'/\sqrt{m}) - (1/\sqrt{m})''/\sqrt{m}] + i\,[2g'/\sqrt{m}]$$

- To solve the spectral problem for $H_+(x)$, we may use well-known method of point-canonical transformation

Mapping to constant-mass problem

$$H_+(x)\psi^+(x) \equiv \left[-\partial\left(\frac{1}{m}\partial\right) + V_+(x) - E^+\right]\psi^+(x) = 0$$

$$\boxed{z = \int^x \sqrt{m(\tau)}d\tau\,, \quad \chi^+(z) = [m(x)]^{-1/4}\,\psi(x)\Big|_{x=z}}$$

$$\left[-\frac{d^2}{dz^2} + \widetilde{V}_+(z) - E^+\right]\chi^+(z) = 0$$

$$\widetilde{V}_+(z) = V^+(x) - \frac{m''(x)}{4m^2(x)} + \frac{7}{16}\frac{m'^2(x)}{m^3(x)}\Big|_{x=z}$$

A simple example for $f = 0$

- In this case $W(x) = i\lambda\sqrt{m(x)}, \quad V_+(x) = -\lambda^2 m(x)$
- Above relation between potential and mass may be exploited to get a solvable problem

For example, choose the following mass function and the superpotential

$$m(x) = \frac{1}{\alpha^2(1+x^2)}, \quad \alpha > 0$$

$$W(x) = i\,\frac{\mu + \alpha/2}{\sqrt{1+x^2}}, \quad \mu > 0$$

Corresponding constant-mass equation

$$\left[-\frac{d^2}{dz^2} + \widetilde{V}_+(z) - \widetilde{E}_n^+\right]\chi_n^+(z) = 0\,, \quad z = \sinh^{-1} x$$

$$\widetilde{V}_+(z) = \frac{\alpha^2}{4} - \mu(\mu+\alpha)\mathrm{sech}^2\alpha z\,,$$

$$\widetilde{E}_n^+(z) = \frac{\alpha^2}{4} - (\mu - n\alpha)^2\,, \quad n \le [\mu/\alpha]$$

$$\chi_n^+(z) = \mathrm{sech}^{\frac{\mu}{\alpha}}(\alpha z) P_n^{(-\frac{\mu}{\alpha}-\frac{1}{2}, -\frac{\mu}{\alpha}-\frac{1}{2})}(i\sinh\alpha z)$$

- Note that the ground state $\chi_0^+(z)$ does not correspond the zero energy state of EM Hamiltonian H_+.
- This implies that both partner Hamiltonians are strictly isospectral, though the SUSY is unbroken due to the existence of both vacuum. This type of peculiarity is known for constant-mass $\mathcal{PT}$-symmetric problems.

Wave functions and spectra of $H_{\pm}(x)$

$$V_{+}(x) = -\frac{(\mu+\alpha/2)^2}{1+x^2}, \quad m(x) = 1/\alpha^2(1+x^2)$$

$$\psi_0^{+}(x) \propto \exp\left[-i\left(\frac{\mu}{\alpha}+\frac{1}{2}\right)\tan^{-1}x\right], E_0^{+} = 0, \quad E_{n+1}^{+} = \frac{\alpha^2}{4} - (\mu - n\alpha)^2$$

$$\psi_{n+1}^{+}(x) \propto (1+x^2)^{-(2\mu+\alpha)/4\alpha} P_n^{(-\frac{\mu}{\alpha}-\frac{1}{2}, -\frac{\mu}{\alpha}-\frac{1}{2})}(ix)$$

Target potential : $V_{-}(x) = -\frac{\mu(\mu+\alpha)+5\alpha^2/4}{1+x^2} - i\alpha\frac{(2\mu+\alpha)x}{1+x^2}$

$$\psi_0^{-}(x) \propto (1+x^2)^{-1/2} \exp\left[i\left(\frac{\mu}{\alpha}+\frac{1}{2}\right)\tan^{-1}x\right], \quad \psi_{n+1}^{-}(x) \propto \mathrm{A}\psi_{n+1}^{+}(x),$$

$$E_n^{-} = E_n^{+}, \qquad \mathrm{A} = \alpha\sqrt{1+x^2}\frac{d}{dx} + i\frac{\mu+\alpha/2}{\sqrt{1+x^2}}$$

Property : $V_{\pm}^{*}(-x) = V_{\pm}(x), \left(\psi_n^{\pm}\right)^{*}(-x) = \psi_n^{\pm}(x)$

We have thus obtained following new class of $\mathcal{PT}$-symmetric potential and mass with real spectra

$$V(x) = -\frac{\eta_1}{1+x^2} - i\eta_2 \frac{x}{1+x^2}\,, \quad m(x) = \frac{1}{\alpha^2(1+x^2)}$$

- Is there any restriction between the parameters for the existence of real spectra?
- To answer this question, it is convenient to look into the corresponding constant-mass $\mathcal{PT}$-symmetric potential

$$\widetilde{V}(z) = -\tilde{\eta}_1 \mathrm{sech}^2 \alpha z - i\tilde{\eta}_2 \mathrm{sech}\alpha z \tanh \alpha z\,, \quad \tilde{\eta}_1 = \eta_1 - \frac{\alpha^2}{4}\,, \tilde{\eta}_2 = \eta_2$$

- Note that for this potential, it is well-known that [Phys. Lett. A 282 (2001) 343] the coupling constants must satisfy following inequality to keep $\mathcal{PT}$-symmetry unbroken

$$|\tilde{\eta}_2| \leq \tilde{\eta}_1 + \frac{\alpha^2}{4}, \; \tilde{\eta}_1 > 0$$

- Hence, for our $\mathcal{PT}$-symmetric EM Hamiltonian, corresponding condition reads

$$|\eta_2| \leq \eta_1 \, , \; \eta_1 > 0$$

Note that for the present example, this restriction is trivially true.

Thank you for your kind attention