Complex Spectrum of a Spontaneously Unbroken PT Symmetric Hamiltonian

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Outline

1. A spontaneously broken Hamiltonian with real spectra
2. Symmetry Property
3. A new approach
4. Conclusions

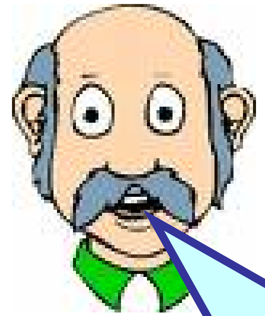
Hamiltonians must be Hermitian

should be replaced by the PT symmetry

PRL 80 5243 (1998)

~1930

1998



there
exist also examples with
real spectra for which
Hamiltonian is not PT
symmetric.

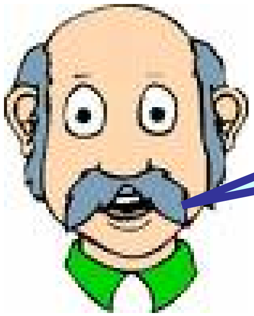
PT Symmetry condition

PT Symmetry

$$p \to p, \quad x \to -x, \quad i \to -i, \quad t \to -t$$

$$[H, PT] = 0 \; ; \qquad PT\Psi(x) = \mp\, \Psi(x)$$

Spontaneous breaking of PT transformations:
The appearances of complex eigenvalues



A Time-dependent Hamiltonian

$$H = p^2 + x^2 + 2if(t)\ x$$

PT Symmetric: $f(-t) = f(t)$

Analytic Solution

$$\Psi_n = \exp\left(-i\Phi(t) + \alpha\, z - \frac{z^2}{2}\right) H_n(z)$$

$$z = x + ig(t)$$

$$\dot{\alpha} = 2(f - g)$$

$$\alpha = \frac{\dot{g}}{2}$$

For n=0, n=1

$$H_0(x + ig) = 1$$

$$H_1(x + ig) = 2(x + ig)$$

C. Yuce, [quant-ph/0703235]
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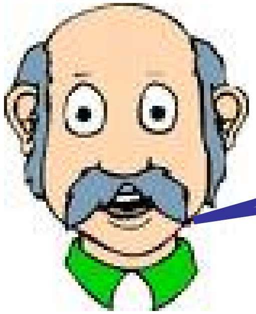
Energy Spectrum

for n=0

$$<0|E|0>=\frac{\int_{-\infty}^{\infty}\Psi_0^{\star}\,H\Psi_0\,dx}{\int_{-\infty}^{\infty}|\Psi_0|^2dx}=1+g^2+\alpha^2+2i\alpha f$$

for n=1

$$<1|E|1>=\frac{E_{11}+2i\alpha f(3+2\alpha^2+2g^2)}{1+2g^2+2\alpha^2}$$



Spectrum is not real unless f α=0

Suppose

$$f(t)=f_0\Rightarrow\alpha(t)=0\quad\left(H=p^2+x^2+2if_0x\right)$$

Examples

$$V(x) = x^2 + 2i\, t^2 x$$

PT Symmetric

PT Symmetric

$$\Psi = \exp\left(t \left(x + i(t^2 - \frac{1}{2}) \right) - \frac{1}{2} \left(x + i(t^2 - \frac{1}{2}) \right)^2 \right)$$

$$\times \quad e^{-2i(n+\frac{1}{2})t - i\,(\frac{t^5}{5} + \frac{t^3}{3} - \frac{t}{4})}\; H_n\left(x + i(t^2 - \frac{1}{2}) \right).$$

$$< 0|E|0 >= \frac{5}{4} + t^4 + 2it^3$$

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Symmetry Property

Theorem: Let $H = \frac{\mathbf{p}^2}{2m} + U^R + iU^I$

The spectrum is real if $< U^I >= \int |\Psi|^2 \, U^I \, d^3x = 0$

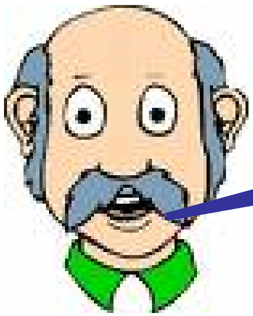
Hermiticity condition

1 $U^I = 0$

generalization of PT condition

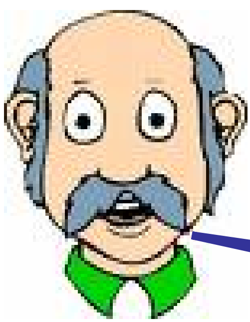
2

$$U^I(-x,t) = -U^I(x,t) \; ;$$
$$|\, \Psi(-x,t) \,|^2 = |\, \Psi(x,t) \,|^2$$



PT condition works only for time-independent potentials

$$
\begin{aligned}
U^{I}(-x,t) &= -U^{I}(x,t)\ ; \\
\mid \Psi(-x,t) \mid^{2} &= \mid \Psi(x,t) \mid^{2}
\end{aligned}
$$



But, they work for both cases.

$$V(x) = x^2 + 2i\, t^2 x$$

PT Symmetric

PT Symmetric

$$\Psi = \exp\left(t\ \left(x + i(t^2 - \frac{1}{2})\right) - \frac{1}{2}\left(x + i(t^2 - \frac{1}{2})\right)^2 \right)$$

$$\times \quad e^{-2i(n+\frac{1}{2})t - i\ (\frac{t^5}{5} + \frac{t^3}{3} - \frac{t}{4})}\ H_n\left(x + i(t^2 - \frac{1}{2})\right).$$

$$U^I(-x,t) = -U^I(x,t)\ ;$$

$$|\ \Psi(-x,t)\ |^2 \neq |\ \Psi(x,t)\ |^2$$

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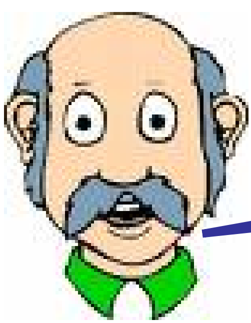
$$H = H_0 + i\lambda V \qquad\qquad V_{nk} =< n|V|k >$$

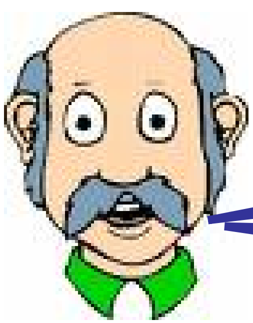
The energy shift:

$$\Delta_n = i\,\lambda V_{nn} - \lambda^2\,\Sigma_{k\neq n}\,\frac{|V_{nk}|^2}{E_n^0 - E_k^0} - i\,\lambda^3$$
$$\left(\Sigma_{k\neq n}\Sigma_{l\neq n}\frac{V_{nk}V_{kl}V_{ln}}{(E_n^0 - E_k^0)(E_n^0 - E_l^0)} - \Sigma_{k\neq n}\frac{|V_{nk}|^2 V_{nn}}{(E_n^0 - E_k^0)^2}\right)$$
$$+ \quad \ldots$$



Spectrum is real if
all of the complex terms vanish!!

- $V_{nn} = 0$
- $V_{nk}V_{kl}V_{ln} = 0$
- $V_{nk}V_{kl}V_{ls}V_{sp}V_{pn} = 0$
- $\ldots$



But, there are infinitely many terms.

Let’s use a and a+ operators.

Examples

$$H = p^2 + x^2 + i\lambda x \qquad x = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right)$$

$$< m|x|n >= \frac{1}{\sqrt{2}}\left(\sqrt{n}\ \delta_{m,n-1} + \sqrt{n+1}\ \delta_{m,n+1}\right)$$

$$V_{nn} = 0 \qquad V_{nk}V_{kl}V_{ln} = 0 \qquad V_{(n\mp 2)n} = 0$$

$$k = n \mp 1 \qquad l = n \mp 2$$

Examples

$$H = p^2 + x^2 + i\lambda x^3$$

$$< m|x^3|n > \neq 0 \qquad if \quad m \mp 1 \quad or \quad m = \mp 3$$

$$V_{nn} = 0 \qquad\qquad V_{nk}V_{kl}V_{ln} = 0$$

$$V_{(n\mp 2)n} = 0$$
$$V_{(n\mp 4)n} = 0$$

$$k = n \mp 1$$
$$k = n \mp 3$$

$$l = n \mp 2$$
$$l = n \mp 4.$$

Examples

$$iV = i\lambda f(\hat{a}) \qquad iV(x) = i\lambda f(\hat{a}^\dagger)$$

$$iV = i\lambda\, e^{\hat{a}^\dagger} = i\lambda\, e^{(x-ip)}$$

$$iV = i\lambda\, \hat{a}^\dagger\ \hat{a}\ \hat{a}^\dagger$$

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Summary of results

- Spontaneously unbroken PT symmetric Hamiltonians may have real spectra.

- New approach: Perturbation theory is useful for searching for a non-Hermitian Hamiltonian with real spectrum.