$\mathcal{PT}$- symmetric waveguide

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Outline:

1. What is $\mathcal{PT}$-symmetry ?
2. Non-Hermitian $\mathcal{PT}$-symmetric waveguide model
3. Spectral analysis

¿ What is $\mathcal{PT}$-symmetry ?

Special case of J-self-adjointness [Edmunds, Evans 1987]

$$H^* = JHJ$$

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$$(\mathcal{P}\psi)(x) := \psi(-x)$$

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N.B.
$$\begin{cases} \sigma_{\mathsf{p}}(H) = \{\lambda \mid H - \lambda \text{ is not injective}\} \\ \sigma_{\mathsf{c}}(H) = \{\lambda \mid H - \lambda \text{ is not surjective} \quad \& \quad \mathfrak{R}(H-\lambda) \text{ is dense}\} \\ \sigma_{\mathsf{r}}(H) = \{\lambda \mid H - \lambda \text{ is injective} \quad \& \quad \mathfrak{R}(H-\lambda) \text{ is not dense}\} \end{cases}$$

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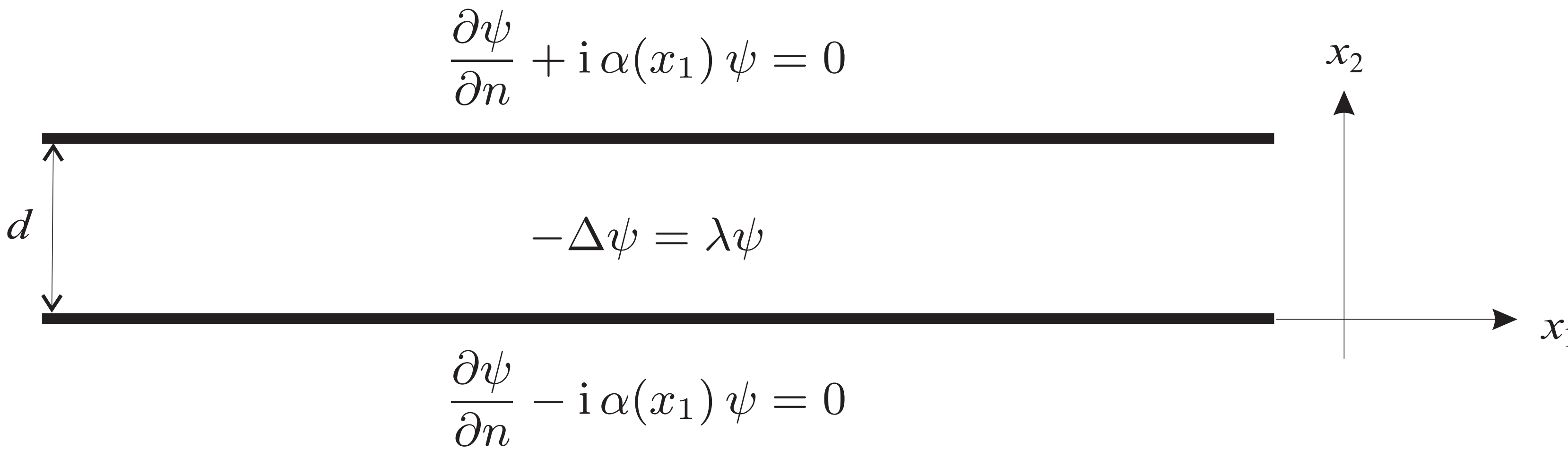
Proof. $\lambda \in \sigma_{\mathsf{r}}(H) \Leftrightarrow \overline{\lambda} \in \sigma_{\mathsf{p}}(H^*) \Leftrightarrow \lambda \in \sigma_{\mathsf{p}}(H)$ *q.e.d.*

$\mathcal{PT}$-symmetric waveguide



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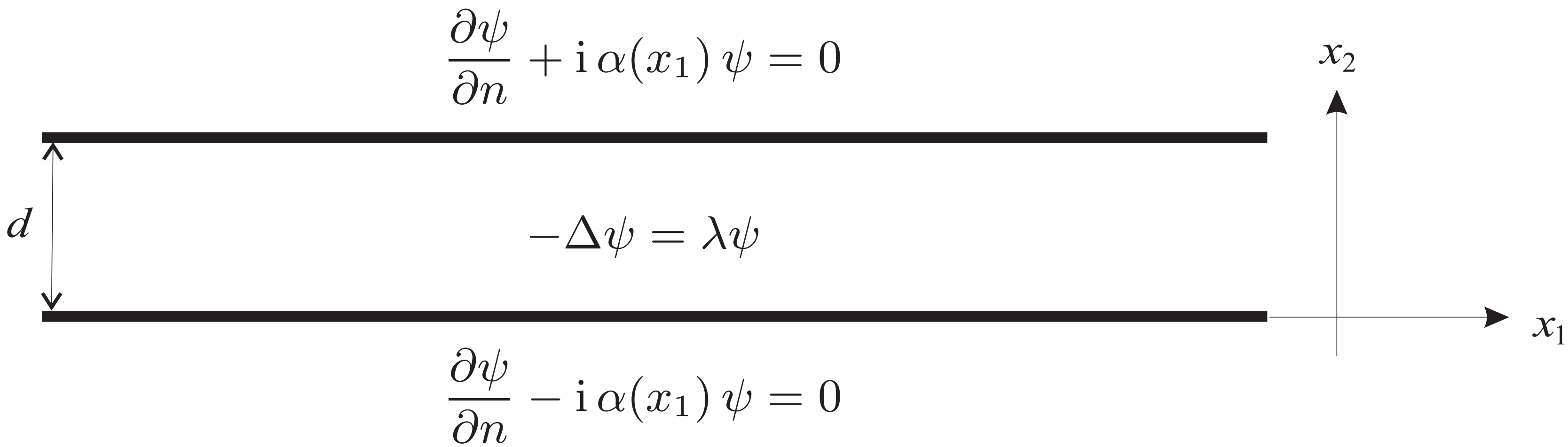

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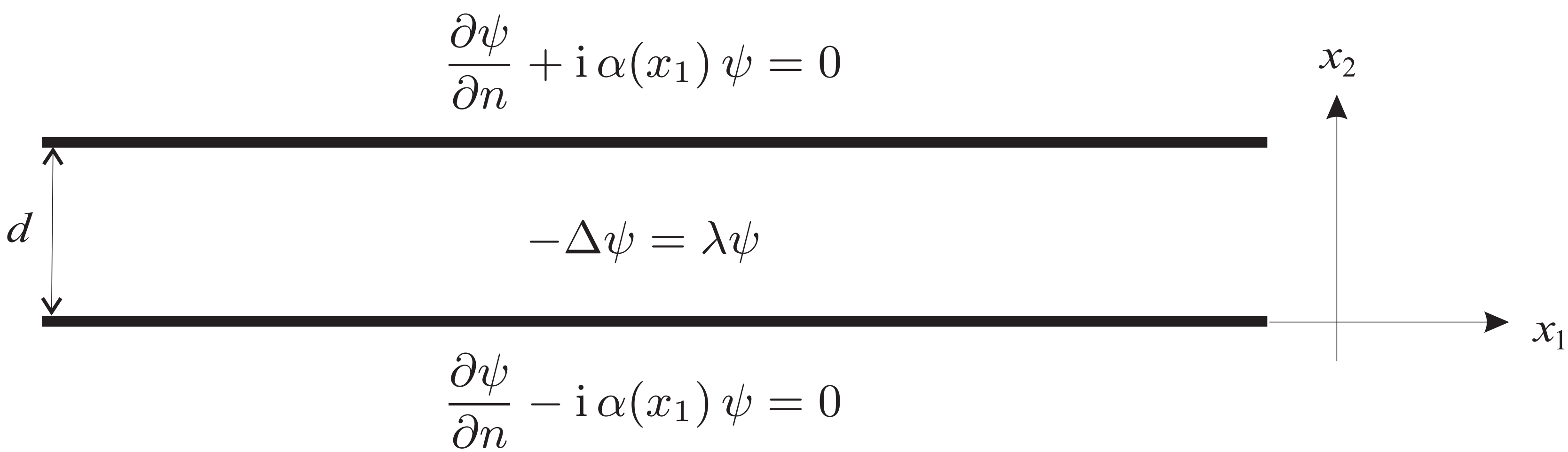

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Theorem. Let $\alpha \in W^{1,\infty}(\mathbb{R})$. Then H_α is an m-sectorial operator satisfying

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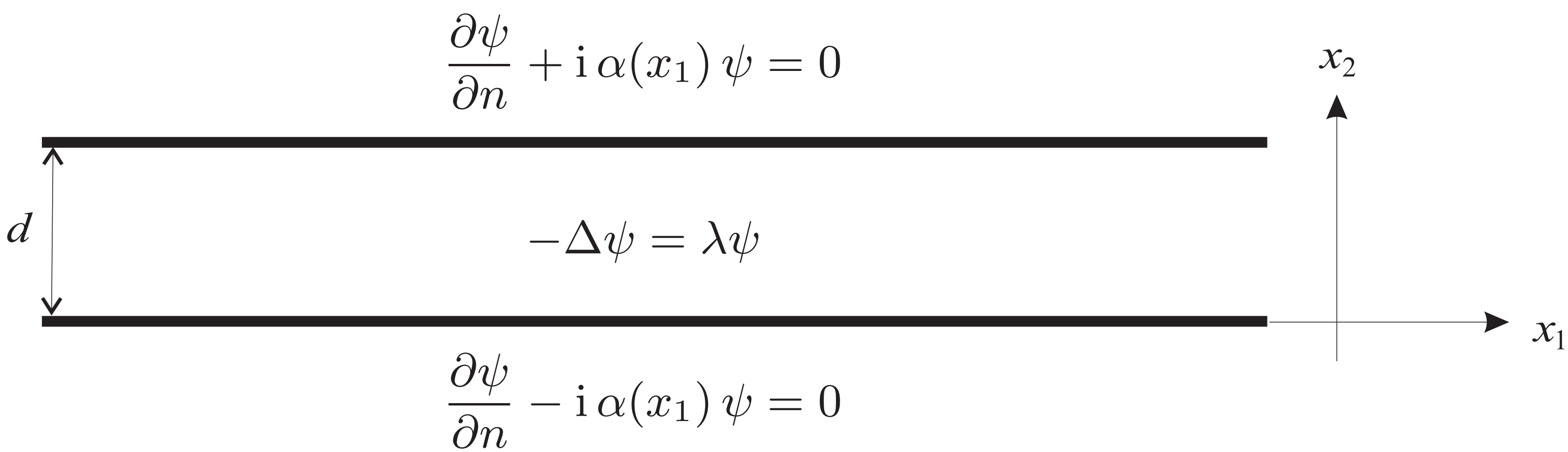

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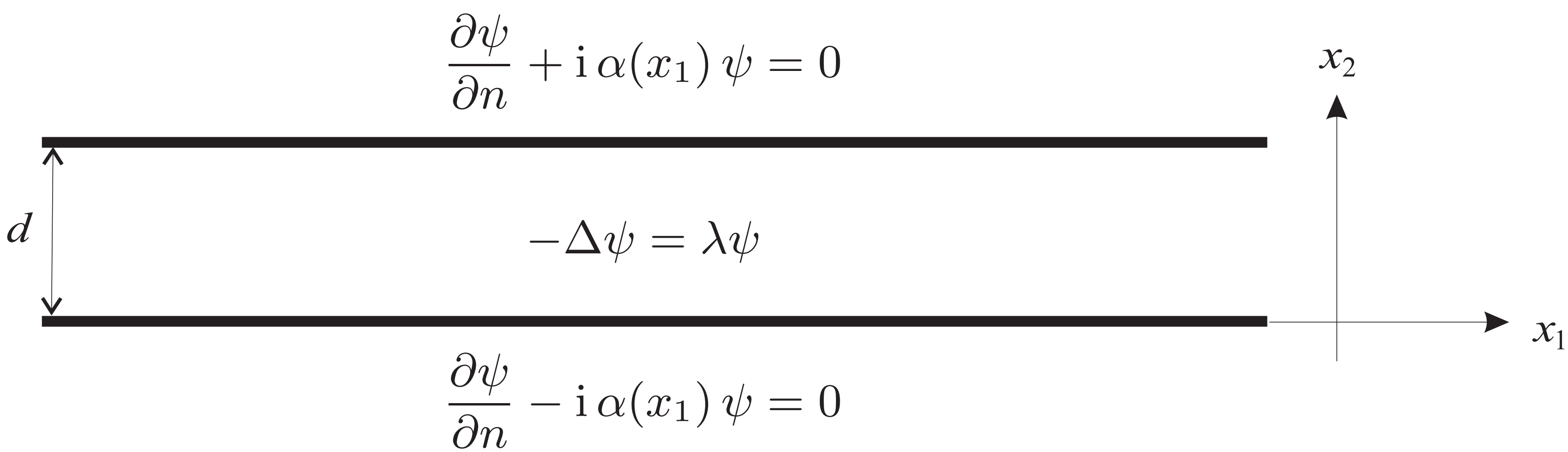

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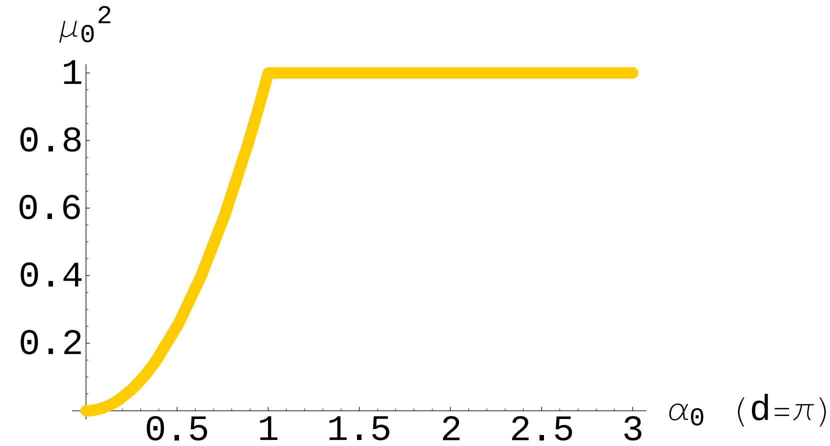
Remark. Schrödinger-type operators in *bounded* domains with non-Hermitian boundary conditions studied by [Kaiser, Neidhardt, Rehberg 2003].

Unperturbed waveguide

$\alpha(x_1) = \alpha_0$

Theorem. $\boxed{\sigma(H_{\alpha_0}) = \sigma_c(H_{\alpha_0}) = \left[\mu_0^2, \infty\right)}$ where $\mu_0 := \min\left\{|\alpha_0|, \pi/d\right\}$.



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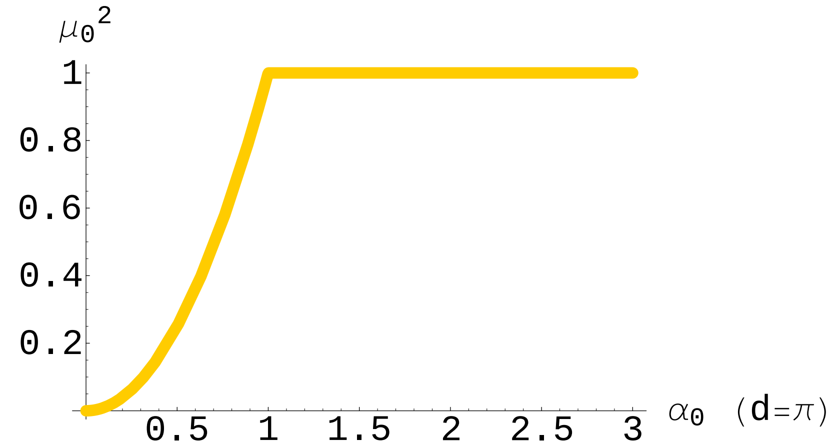

Proof ("separation of variables").

$$\sigma(H_{\alpha_0}) = \sigma(-\Delta^{\mathbb{R}}) + \sigma(-\Delta_{\alpha_0}^{(0,d)})$$

$$= [0, \infty) + \{\alpha_0^2\} \cup \left\{(j\pi/d)^2\right\}_{j=1}^{\infty}$$

[D.K., Bíla, Znojil 2006]

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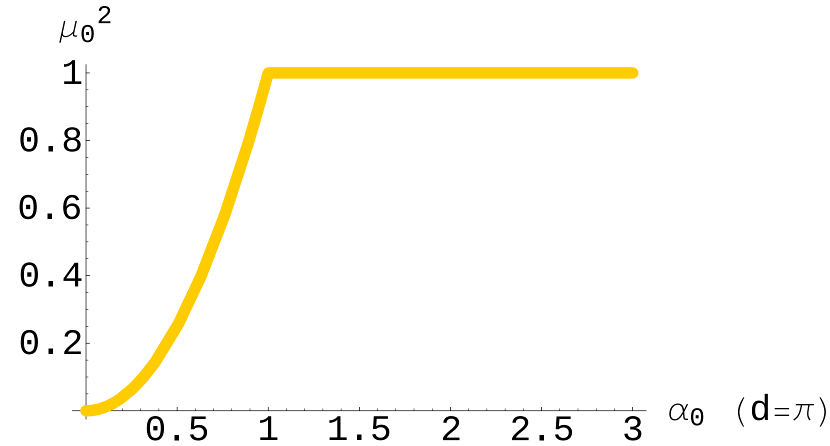

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Notation. $\mu_1 := \max\left\{|\alpha_0|, \pi/d\right\}$, $\mu_j := j\pi/d$ for $j \geq 2$

$$\psi_j^{\alpha_0}(x_2) := \cos(\mu_j x_2) - \mathrm{i}\,\frac{\alpha_0}{\mu_j}\,\sin(\mu_j x_2)$$

Stability of the continuous spectrum

Theorem. Let $\alpha - \alpha_0 \in C_0(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. Then $\sigma_{\mathsf{c}}(H_\alpha) = \big[\mu_0^2, \infty\big)$.

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Theorem. Let $\alpha \in C_0(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ be odd. Then $\sigma_p(H_\alpha) \subset \mathbb{R}$.

Weakly-coupled bound states

$$\alpha(x_1) = \alpha_0 + \varepsilon\,\beta(x_1)$$

$\varepsilon \longrightarrow 0+$ $\quad \beta \in C_0^2(\mathbb{R})$

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Theorem($|\alpha_0| < \pi/d$).

1. $\alpha_0 \langle \beta \rangle < 0 \implies \exists!\ \lambda_\varepsilon = \mu_0^2 - \varepsilon^2 \alpha_0^2 \langle \beta \rangle^2 + 2\varepsilon^3 \alpha_0 \langle \beta \rangle \tau + \mathcal{O}(\varepsilon^4) \ \in \mathbb{R}$
2. $\alpha_0 \langle \beta \rangle > 0 \implies$ no
3. $\alpha_0 = 0 \implies$ no
4. $\langle \beta \rangle = 0$ & $\tau > 0 \implies \exists!\ \lambda_\varepsilon = \mu_0^2 - \varepsilon^4 \tau^2 + \mathcal{O}(\varepsilon^5) \ \in \mathbb{R}$
5. $\langle \beta \rangle = 0$ & $\tau < 0 \implies$ no

Theorem($|\alpha_0| > \pi/d$).

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$$\langle \beta \rangle := \int_{\mathbb{R}} \beta(x_1)\, \mathrm{d}x_1$$

The mysterious τ

$$\tau = \tau(\beta, d, \alpha_0)$$

$$\tau := \begin{cases} 2\alpha_0^2\langle\beta v_0\rangle + \dfrac{2\alpha_0}{d}\displaystyle\sum_{j=1}^{\infty}\dfrac{\mu_j^2\langle\beta v_j\rangle}{\mu_j^2-\mu_0^2}\,\tan\dfrac{\alpha_0 d + j\pi}{2} & \text{if} \quad |\alpha_0| < \dfrac{\pi}{d} \\ \dfrac{2\alpha_0\pi^2\cot\frac{\alpha_0 d}{2}}{(\mu_1^2-\mu_0^2)d^3}\langle\beta v_1\rangle + \dfrac{8\pi^2}{(\mu_1^2-\mu_0^2)d^4}\displaystyle\sum_{j=1}^{\infty}\dfrac{\mu_{2j}^2\langle\beta v_{2j}\rangle}{\mu_{2j}^2-\mu_0^2} & \text{if} \quad |\alpha_0| > \dfrac{\pi}{d} \end{cases}$$

where

$$v_j(x_1) := \begin{cases} -\dfrac{1}{2}\displaystyle\int_{\mathbb{R}} |x_1 - x_1'|\,\beta(x_1')\,\mathrm{d}x_1' & \text{if} \quad j = 0 \\ \dfrac{1}{2\sqrt{\mu_j^2-\mu_0^2}}\displaystyle\int_{\mathbb{R}} e^{-\sqrt{\mu_j^2-\mu_0^2}\,|x_1-x_1'|}\,\beta(x_1')\,\mathrm{d}x_1' & \text{if} \quad j \geqslant 1 \end{cases}$$

are the solutions to $-v_j'' + (\mu_j^2 - \mu_0^2)v_j = \beta$ in $\mathbb{R}$.

Proposition. If $0 < \alpha_0 < \pi/d$, $\langle\beta\rangle = 0$ and $\mathrm{supp}\beta$ is wide enough, then $\tau > 0$.

Eigenfunction asymptotics

Theorem. If a weakly-coupled eigenvalue λ_ε exists, the associated eigenfunction Ψ_ε can be chosen so that it satisfies

1. $\boxed{\Psi_\varepsilon(x) = \psi_0^{\alpha_0}(x_2) + \mathcal{O}(\varepsilon)}$ in $W^{2,2}\big(\Omega \cap \{x : |x_1| < a\}\big)$ for each $a > 0$,

2. $\boxed{\Psi_\varepsilon(x) = e^{-\sqrt{\mu_0^2-\lambda_\varepsilon}|x_1|}\,\psi_0^{\alpha_0}(x_2) + \mathcal{O}(e^{-\sqrt{\mu_0^2-\lambda_\varepsilon}|x_1|})}$ as $|x_1| \to \infty$.

Conclusions

Summary :

$\longrightarrow$ non-Hermitian $\mathcal{PT}$-symmetric model with both the point and continuous spectra

$\longrightarrow$ the residual spectrum is empty

$\longrightarrow$ the continuous spectrum is real

$\longrightarrow$ the weakly-coupled eigenvalues are real

$\longrightarrow$ the total spectrum is real under additional symmetry condition

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Open problems :

¿ reality of the spectrum without the additional symmetry condition ?

¿ non-perturbative proof of the existence of the point spectrum ?

¿ numerics ?

¿ phenomenological relevance ?