The metric in quasi-Hermitian quantum mechanics: overview and recent results

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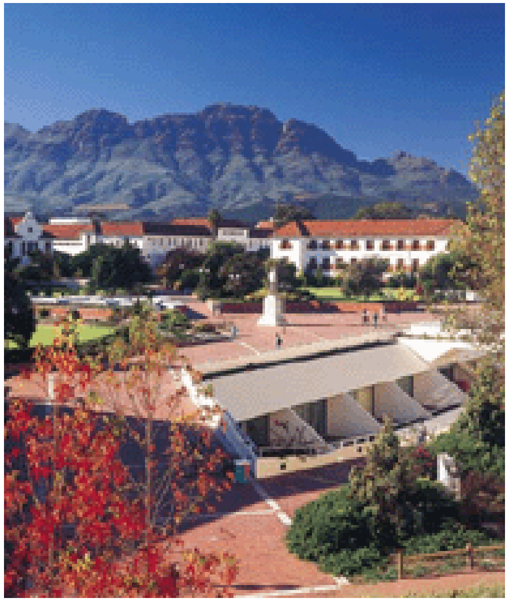
Central campus with underground library



Twin peaks with snow

(1504 m)

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Stellenbosch Vineyards

Stellenbosch (founded 1673)

Historic Stellenbosch – Cape Dutch Style

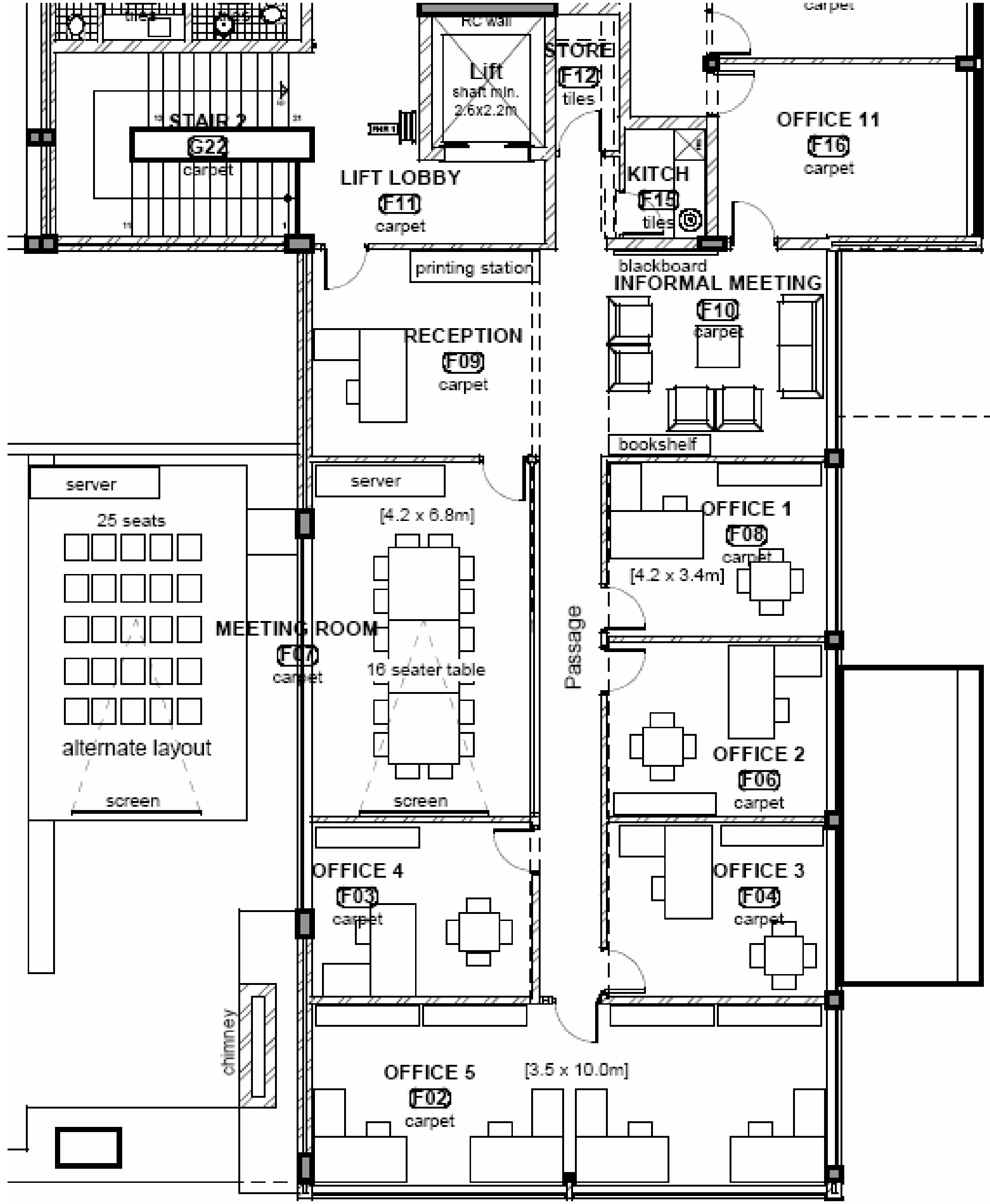
Stellenbosch

Historical Mostertsdrift Homestead
Stellenbosch Institute for Advanced Study (STIAS)

National Institute for Theoretical Physics (NITheP)

at

Stellenbosch Institute for Advanced Study (STIAS)



View from STIAS/NITheP office

View from postdoc office

View from postdoc office (overlooking vineyard area)

STIAS offices – to be used by NITheP for 6-8 weeks programmes

STIAS seminar area (up to 150 people; can be subdivided)

Brief outline

- Non-Hermitian Hamiltonians in the context of interacting boson models
- General framework for ***consistent*** non-Hermitian QM
- Framework of $\mathcal{PT}$-symmetric QM – links to above
- Role and construction of the metric – Moyal products
- The non-Hermitian oscillator: an example
- Possible link to Berry connection and curvature; ground state phase information in the metric?
- Conclusions; avenues to explore

Non-Hermitian Hamiltonians in the context of interacting boson models

On microscopic level arise through application of the non-unitary Dyson-type mapping to bifermion operators (schematically)

$$c^\dagger c^\dagger \longleftrightarrow f(B^\dagger, B) = B^\dagger - B^\dagger B^\dagger B$$

$$cc \longleftrightarrow g(B^\dagger, B) = B$$

$$c^\dagger c \longleftrightarrow h(B^\dagger, B) = B^\dagger B$$

$$g \neq f^\dagger$$

A (Hermitian) 1-plus-2-body fermion Hamiltonian is generally mapped onto a non-Hermitian 1-plus-2-body boson Hamiltonian
In the boson Hamiltonian this typically leads to terms of the type

$$\alpha B_i^\dagger B_j^\dagger B_k B_l + \beta B_l^\dagger B_k^\dagger B_j B_i$$

$$\alpha \neq \beta$$

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Consider the following two possible (Holstein-Primakoff and Dyson) boson realisations of SU(2) fermion pair operators

$$J_+ = \sum_{m=1}^{\Omega} a_m^\dagger a_{-m}^\dagger \rightarrow b^\dagger \sqrt{2\Omega - b^\dagger b} \rightarrow b^\dagger \left(2\Omega - b^\dagger b\right)$$

$$J_- = \sum_{m=1}^{\Omega} a_{-m} a_m \rightarrow \sqrt{2\Omega - b^\dagger b}\; b \rightarrow b$$

$$J_z = \sum_{m=1}^{\Omega} a_m^\dagger a_m \rightarrow b^\dagger b - \Omega \rightarrow b^\dagger b - \Omega$$

The pairing Hamiltonian $H = J_+ J_-$ maps onto an

Hermitian boson Hamiltonian in both cases, but not so for eg

$$H = J_+ J_+ + J_- J_-$$

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Since the mapping is faithful (all algebraic properties are preserved), it is here guaranteed that the spectrum of the non-Hermitian Hamiltonian will be real (and identical to the original spectrum)

Caveat of physical subspace

Question: Can a criterion be given for a *general* (eg phenomenological) non-Hermitian Hamiltonian to have a real spectrum?

If so, can a consistent quantum mechanical framework be constructed on this basis?

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Answer is positive

FG Scholtz, HB Geyer & FJW Hahne
Ann Phys (NY) 213 (1992) 74-101

Require existence of a linear operator (metric) Θ on Hilbert space $\mathcal{H}$

$\Theta : \mathcal{H} \rightarrow \mathcal{H}$ such that

(i) $\mathcal{D}(\Theta) = \mathcal{H}$

(ii) $\Theta^{\dagger} = \Theta$ (Hermiticity)

(iii) $\left(\varphi, \Theta\varphi\right) > 0 \;\; \forall \;\; \varphi \in \mathcal{H}$ and $\varphi \neq 0$ (positive definiteness)

(iv) $\left\|\Theta\varphi\right\| \leq \left\|\Theta\right\| \left\|\varphi\right\| \;\; \forall \;\; \varphi \in \mathcal{H}$ (boundedness)

(v) $\Theta H = H^{\dagger}\Theta$ (H is quasi-Hermitian wrt metric Θ)

A note on terminology:

“Quasi-Hermitian” was introduced in our 1992 Ann Phys paper, following existing terminology in linear algebra (eg “Methods of Matrix Algebra” by MC Pease III (NY, Academic, 1965)), now refering to a complete and consistent framework for non-Hermitian QM.

In papers since 2002 by Mostafazadeh (and other authors) “pseudo-Hermitian” has been used for the same concept, although without the requirement of positivity for the metric, since the primary focus (at first) was on conditions for the reality of the spectrum of a non-Hermitian Hamiltonian, for which the existence of Θ with $\Theta H = H^{\dagger}\Theta$ is sufficient.

The metric Θ is not uniquely defined by these conditions.

However, by requiring

$$\Theta A_i = A_i^{\dagger}\Theta \quad \forall i$$

for a set of operators A_i which is irreducible (and includes H),

uniqueness can be proved

The introduction of the metric Θ amounts to the introduction of a modified inner product

$$(\varphi,\psi)_{\Theta} \equiv (\varphi,\Theta\psi)$$

$$\Rightarrow (\varphi,A_i\psi)_{\Theta} = (\varphi,\Theta A_i\psi) = (\varphi, A_i^{\dagger}\Theta\psi) = (A_i\varphi,\psi)_{\Theta}$$

In some sense the metric fixes the physical content of the theory. What does this mean and how can it be used…?

Link to Gauge Theories

- In Dirac quantisation of a gauge theory, the physical Hilbert space is defined as the subspace annihilated by the (first class) constraints.
- What is the inner product on the physical Hilbert space?
- Ashtekar and Rendall considered this issue in parallel with our work (1992).
- Conclusion: if the gauge invariant observables (which commute weakly with the constraints) form an irreducible set, the inner product on the physical Hilbert space is uniquely determined.
- Again, the observables dictate the choice of inner product (Hilbert space).

PT-symmetric quantum mechanics

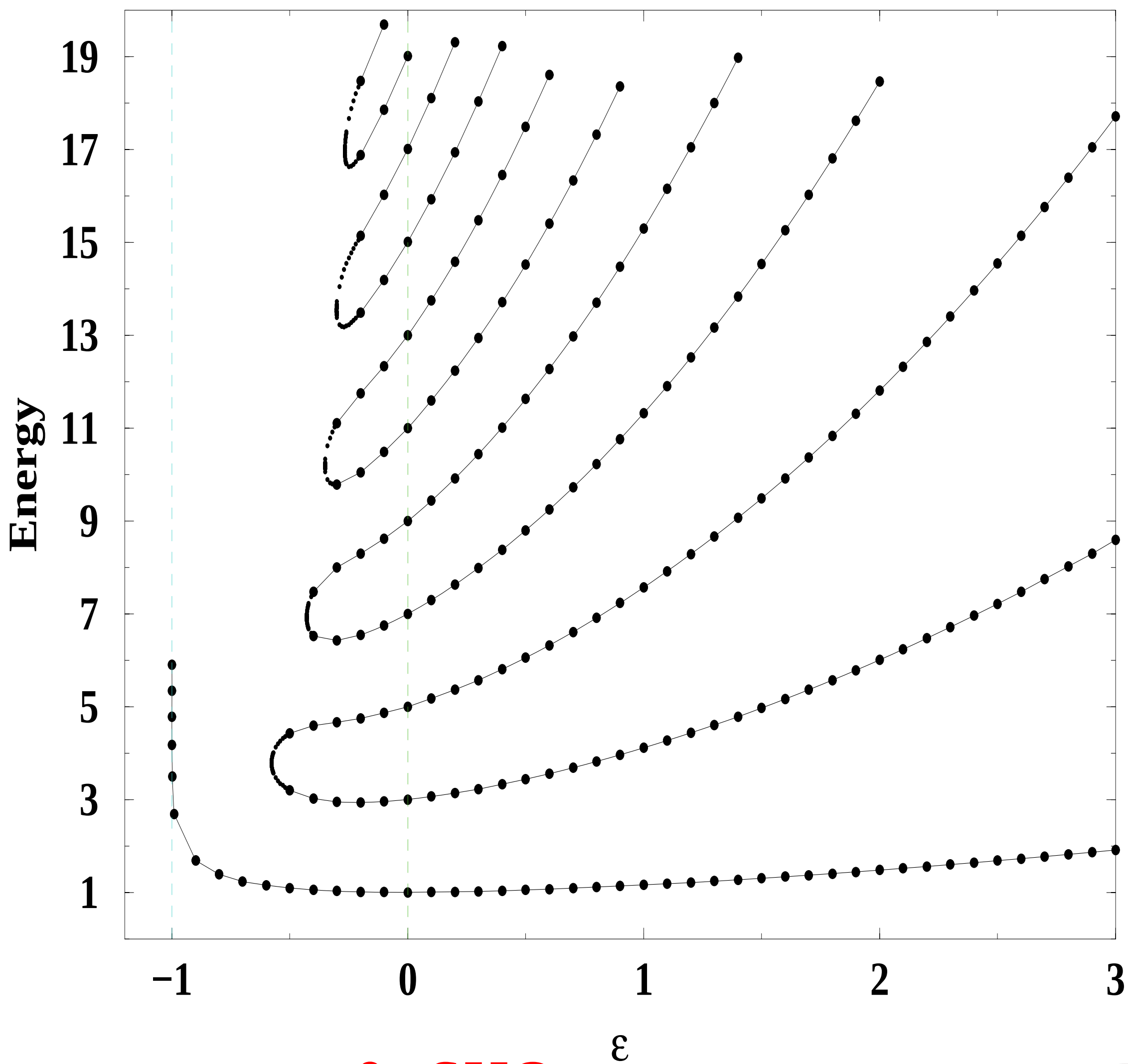
Carl Bender *et al*

Developed from studies of the class of Hamiltonians

$$H = p^2 + x^2 (ix)^\varepsilon$$

for which numerical studies (based on, and supported by, in-depth analysis) confirmed a real spectrum only for $\varepsilon \geq 0$ (subsequently strictly proven by Dorey *et al* via Bethe ansatz).

PT Symmetry – "trademark cartoon" (from Bender *et al*)



$H = p^2 + x^2(ix)^{\varepsilon}$

$\varepsilon \geq 0$

spectrum real, positive

$-1 < \varepsilon < 0$

pos eigenvalues: finite #

complex conjugate pairs

$\varepsilon = 0$, SHO

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Emphasised by Bender *et al* that the reality of the spectrum may be linked to $\mathcal{PT}$-symmetry (ie invariance of *H* under simultaneous parity and time reversal)

The parity operator $\mathcal{P}$ is *linear*

$$p \rightarrow -p \quad \text{and} \quad x \rightarrow -x$$

The time-reversal operator $\mathcal{T}$ is *anti - linear*

$$p \rightarrow -p, \quad x \rightarrow x \;\; \text{and} \quad i \rightarrow -i$$

For *unbroken* $\mathcal{PT}$-symmetry (simultaneous eigenstate of *H* and $\mathcal{PT}$) reality follows readily. However, $[H, \mathcal{PT}]=0$ does *not* generally imply simultaneous eigenstates, since $\mathcal{PT}$ is anti-linear. Assumption is non-trivial, as it is not simple to determine *a priori* whether $\mathcal{PT}$-symmetry is unbroken.

Link to previous considerations (metric) by introducing the so-called $\mathcal{C}$-operator

Properties similar to standard charge operator $C^2 = 1$

Position space representation $C\phi_n(x) = (-1)^n \phi_n(x)$

$$C(x,y) = \sum_n \phi_n(x)\phi_n(y), \text{ where the } \phi_n(x) \text{ are eigenstates of } H$$

Introduce a modified inner product

$$\langle f | g \rangle_{CPT} \equiv \int_C dx \left[\mathcal{CPT} f(x) \right] g(x)$$

with completeness relation

$$\sum_n \phi_n(x) \left[\mathcal{CPT} \phi_n(y) \right] = \delta(x-y)$$

This inner product is positive definite, dynamically determined by *H*

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Three stages in the development of PT-symmetric QM

- Real spectra for (some) non-Hermitian Hamiltonians
- Link with $\mathcal{PT}$-symmetry
- Identification of a positive definite inner product

$\Rightarrow$ consistent QM framework

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Role and construction of the metric – Moyal products

FG Scholtz & HBG, PLB 634 (2006) 84
J Phys A 39 (2006) 10189

Constructing the metric Θ, it is required to solve the operator equation

$$\Theta H = H^\dagger \Theta, \quad \text{where } \Theta = \Theta(x, p)$$

Exploit the Moyal construction which re-writes the *operator* equation as a standard *partial differential* equation, based on the Moyal or star product (replacing the ordinary product)

$$A(x,p) * B(x,p) \equiv A(x,p) e^{i\hbar \overleftarrow{\partial}_x \overrightarrow{\partial}_p} B(x,p)$$

where the non-commutative nature of *x* and *p* is captured by directional derivatives acting on ordinary functions

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Check: suppose we specify $\hat{x}$ and $\hat{p}$ to be observables (other observables such as *H* are to be functions of $\hat{x}$ and $\hat{p}$), then the equations for the metric are

$$\left.\begin{aligned}\Theta\hat{x} &= \hat{x}\Theta \\ \Theta\hat{p} &= \hat{p}\Theta\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\Theta * x &= x * \Theta \\ \Theta * p &= p * \Theta\end{aligned}\right\} \Rightarrow \frac{\partial\Theta}{\partial x} = \frac{\partial\Theta}{\partial p} = 0 \Rightarrow \Theta = \text{const.}$$

as in standard QM.

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Moyal products – brief background

For Hilbert space with finite dimension *N*, construct unitary irrep of Heisenberg-Weyl algebra

$$gh = e^{i\phi} hg\,; \quad g^{\dagger} = g^{-1},\ h^{\dagger} = h^{-1}\,,$$

$$\phi = 2\pi / N$$

$$U(n,m) = g^n h^m \text{ forms a basis, } m,n = 0,1 \ldots\ldots N-1$$

$$\text{Expand any operator } A = \sum_{n,m=0}^{N-1} a_{n,m}\, g^n h^m\,, \quad a_{n,m} = \left(U(n,m),A\right)/N$$

$$\text{with } (B,A) \equiv \operatorname{tr} B^{\dagger} A$$

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Substitute $g \to e^{i\alpha}\ ,\ h \to e^{i\beta};\ \alpha,\beta \in [\,0, 2\pi)$

turns *A* into a function $A(\alpha,\beta)=\sum_{n,m=0}^{N-1} a_{n,m} e^{in\alpha} e^{im\beta}$

uniquely detemined by the operator *A.*

Isomorphism with operator product *AB* now established by Moyal or star product

$$A(\alpha,\beta) * B(\alpha,\beta) \equiv A(\alpha,\beta) e^{i\phi\, \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta} B(\alpha,\beta)$$

where directional derivatives in the exponent capture the non-commutative nature of the operators

Given the function

$A(\alpha,\beta)$ establish the coefficients $a_{n,m}$

through Fourier transformation, and finally the *operator A*

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Can establish a relation between the two *functions* which represent a given *operator* and its Hermitian conjugate.

This can then be used to establish the condition for Hermiticity on the level of *functions*

$$A^*(\alpha,\beta) = e^{-i\phi\,\partial_\alpha\partial_\beta} A(\alpha,\beta).$$

All of these results for a finite dimensional Hilbert space can be generalized to the case of QM.

The main result is the form of the Moyal product which now reads

$$A(x,p) * B(x,p) \equiv A(x,p) e^{i\hbar\,\overleftarrow{\partial}_x \overrightarrow{\partial}_p} B(x,p)$$

A shifted oscillator – the *ix* potential

The shifted harmonic oscillator with

$V(x) = \frac{1}{2}x^2 + \gamma x$ can of course be solved exactly,

also for the $\mathcal{PT}$-symmetric case $\gamma = i$

It is also known that the $\mathcal{C}$-operator and the metric Θ can be related by

$$\mathcal{C} = \Theta^{-1}\mathcal{P}$$

in this case the *C*-operator had been solved (Bender) as

$$\mathcal{C} = e^{-2p}\mathcal{P}$$

From the Moyal product construction the metric Θ is solved from the PDE

$$2ix\Theta(x,p) + (ix-1)\Theta^{(0,1)} - \tfrac{1}{2}\Theta^{(0,2)} - i\Theta^{(1,0)}p + \tfrac{1}{2}\Theta^{(2,0)} = 0,$$

$$\text{with}\quad \Theta^{(m,n)} = \frac{\partial^{m+n}\Theta}{\partial^m x\,\partial^n p}$$

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Assuming $\Theta = \Theta(p)$, the PDE reduces to the ODE

$$2ix\Theta + (ix-1)\Theta' - \tfrac{1}{2}\Theta'' = 0,$$

with solution $\Theta = e^{-2p}$ as before

From here all the standard results for the shifted oscillator can be obtained

Hermiticity and positive definitenesss of the metric Θ

The PDE for the metric Θ is linear, of the form $L\Theta(x,p) = 0$. From

$$e^{-i\hbar\partial_x\partial_p}\, x\, e^{i\hbar\partial_x\partial_p} = x - i\hbar\partial_p$$

and

$$e^{-i\hbar\partial_x\partial_p}\, p\, e^{i\hbar\partial_x\partial_p} = p - i\hbar\partial_x$$

it follows that

$$e^{-i\hbar\partial_x\partial_p}\, L\, e^{i\hbar\partial_x\partial_p} = -L^*$$

implying

$$L^*\, e^{-i\hbar\partial_x\partial_p}\Theta(x,p) = 0.$$

But

$$L^*\,\Theta^*(x,p) = 0.$$

Thus, provided the boundary conditions also satisfy the general hermiticity condition

$$A^*(x,p) = e^{-i\hbar\partial_x\partial_p} A(x,p)\,,$$

then

$$\Theta^*(x,p) = e^{-i\hbar\partial_x\partial_p}\Theta(x,p)$$

ie the metric is guaranteed to be Hermitian, since L is linear (and has a unique solution).

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For the shifted oscillator e.g. this follows trivially; for the real metric

$$\Theta = e^{-2p}$$

which is a function of *p* only,

$$e^{-i\hbar \partial_x \partial_p} \Theta(x,p) = \Theta(p) = \Theta^*(p),$$

i.e. Hermitian.

To verify positive definiteness, one generally verifies that the logarithm of the metric is Hermitian. First requires that the *function* corresponding to the logarithm has to be found, ie find η*(x,p)* such that

$$\Theta = 1 + \eta + \frac{1}{2!}\eta * \eta + \frac{1}{3!}\eta * \eta * \eta + ...$$

Here the Moyal product trivially reduces to an ordinary product, the logarithm of *Θ* is simply *-2p,* and again obviously Hermitian.

Example: non-Hermitian oscillator

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) + \alpha a^2 + \beta a^{\dagger 2}$$

$$\alpha \neq \beta$$

Can solve this by rescaling $a \to \lambda a$ and $a^\dagger \to \lambda^{-1} a^\dagger$

$$\tilde{H} = \omega\left(a^\dagger a + \frac{1}{2}\right) + \sqrt{\alpha\beta}\left(a^2 + a^{\dagger 2}\right)$$

Diagonalise with standard Bogoliubov transformation;

Yields SHO, effective frequency $\Omega = \sqrt{\omega^2 - 4\alpha\beta}$

Spectrum $E_n = (n + 1/2)\hbar\Omega$

If S hermitizes H, then H is quasi-Hermitian wrt $\Theta = S^\dagger S$

Here $S = \left(\frac{\alpha}{\beta}\right)^{\hat{n}/4}$ (with $\hat{n} = a^\dagger a$)

$$\Rightarrow \quad \tilde{H} = SHS^{-1} = \tilde{H}^\dagger$$

Choosing different observables to complete the irreducible set together with the Hamiltonian yields different metrics

$$\text{number operator } \hat{n} \;\Rightarrow\; \Theta(\hat{n}) = \left(\frac{\alpha}{\beta}\right)^{\hat{n}/2}$$

$$\text{position } x \Rightarrow \Theta(x) = \exp\left(\frac{\alpha-\beta}{(\omega-\alpha-\beta)}x^2\right)$$

$$\text{momentum } p \Rightarrow \Theta(p) = \exp\left(-\frac{\alpha-\beta}{(\omega+\alpha+\beta)}p^2\right)$$

These can be obtained by solving simple difference or differential equations

Using the Moyal construction to obtain Θ in general, first re-write $\hat{H}\left(a^\dagger, a\right) = \hat{H}\left(x, p\right)$

$$\hat{H} - \omega/2 = a\hat{p}^2 + b\hat{x}^2 + ic\hat{p}\hat{x}$$

$$a = \left(\omega - \alpha - \beta\right)/2,\ b = \left(\omega + \alpha + \beta\right)/2,\ c = \left(\alpha - \beta\right)$$

This yields the associated functions (*p* ordered to left of *x* at operator level)

$$H\left(x,p\right) = ap^2 + bx^2 + icpx; \quad H^\dagger\left(x,p\right) ap^2 + bx^2 - icpx + c$$

From $H\left(x,p\right) * \Theta\left(x,p\right) = \Theta\left(x,p\right) * H^\dagger\left(x,p\right)$ one

finds the PDE

$$c(1-2\,i\,p\,x)\,\Theta(x,p)+\,(c\,p\,-\,2\,i\,b\,x)\Theta^{(0,1)}(x,p)+$$
$$(c\,x\,+\,2\,i\,a\,p)\Theta^{(1,0)}(x,p)+b\,\Theta^{(0,2)}(x,p)-\,a\,\Theta^{(0,2)}(x,p)=0$$

where $\Theta^{(m,n)}=\dfrac{\partial^{n+m}\Theta}{\partial^{n}x\,\partial^{m}p}$

Choice of boundary conditions
$\leftrightarrow$ non-uniqueness of metric

General solution is $\Theta(x,p)=\exp\,(rp^2+spx+tx^2)$

with *s* a free parameter, and

$$r=\frac{-c\pm\sqrt{c^2-4\,a\,b\,\hbar s(2i-\hbar s)}}{4\,b\hbar}\;;\;t=\frac{c\pm\sqrt{c^2-4\,a\,b\hbar\,s(2i-\hbar s)}}{4\,a\,\hbar}$$

(essential singularity at $\hbar=0$; metric not a classical object)

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Specifying *p* as an observable (in addition to *H*) requires

$$p * \Theta(x,p) = \Theta(x,p) * p$$ which gives $$\Theta^{(1,0)}(x,p) = 0$$

i.e. $$\Theta(x,p) = \Theta(p)$$

This requires $$s = 0;\ t = 0;\ r = -\frac{c}{2b} = -\frac{\alpha - \beta}{\omega + \alpha + \beta}$$

with $$\Theta(p) = \exp\left(-\frac{\alpha - \beta}{(\omega + \alpha + \beta)} p^2\right)$$ as before

One can now continue to calculate matrix elements of various physical quantities, recalling that $S = \sqrt{\Theta}$ hermitizes H and that the inverse transformation can be used to obtain operators X and P which could be viewed as equivalent to x and p , viz

$$X = S^{-1} x S$$
$$P = S^{-1} p S$$

$$S^2 = \Theta$$

and using the modified inner product

$$(\varphi,\psi)_\Theta \equiv (\varphi,\Theta\psi)$$

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Also gain insight into the problem by starting directly from an *ansatz* for the similarity transformation that hermitizes *H* :

$$S = \exp A$$

$$A = \varepsilon a^\dagger a + \eta a^2 + \eta^* a^{\dagger 2}$$

$$SaS^{-1} = (\cosh\theta - \frac{\varepsilon}{\theta}\sinh\theta)a - 2\frac{\eta^*}{\theta}\sinh\theta a^\dagger$$

$$Sa^\dagger S^{-1} = (\cosh\theta + \frac{\varepsilon}{\theta}\sinh\theta)a^\dagger + 2\frac{\eta^*}{\theta}\sinh\theta a$$

$$\theta = \sqrt{\varepsilon^2 - 4|\eta|^2}$$

$$h_S = SHS^{-1} = U(\varepsilon,\eta)\left(a^\dagger a + \frac{1}{2}\right) + V(\varepsilon,\eta)a^2 + W(\varepsilon,\eta)a^{\dagger 2}$$

h_S Hermitian requires

$$U \in \mathbb{R};\; V = W^{*}, \text{ leading to}$$

$$\frac{\tanh\theta}{\theta} = \frac{\alpha-\beta}{(\alpha+\beta)\varepsilon - 2\omega\eta} \quad \text{with } \eta = \eta^{*}$$

Position and momentum observables are accordingly given by

$$X = S^{-1}\hat{x}S = (\cosh\theta\,\hat{x} + \frac{i}{\omega}\frac{\varepsilon-2\eta}{\theta}\sinh\theta\,\hat{p}$$

$$P = S^{-1}\hat{p}S = (\cosh\theta\,\hat{p} - i\omega\frac{\varepsilon+2\eta}{\theta}\sinh\theta\,\hat{x}$$

Clearly metric dependent

$$\varepsilon = \frac{1}{2\sqrt{1-z^{2}}}\operatorname{arctanh}\frac{(\alpha-\beta)\sqrt{1-z^{2}}}{\alpha+\beta-z\omega}$$

Define $z = \varepsilon / 2\eta$

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$$h_{S(z)} = \frac{1}{2}\left(\mu(z)\hat{p}^2 + \nu(z)\hat{x}^2\right)$$

$\mu(z)$, $\nu(z)$ functions of α, β, ω and z

S(*z*) is obtained similarly

For *z=0* one has $\varepsilon = 1/4 \ln(\alpha/\beta)$

$$\Theta = S^2 = \left(\frac{\alpha}{\beta}\right)^{\hat{n}/2}$$

and

$$h_{S(z=0)} = \frac{\omega - 2\sqrt{\alpha\beta}}{2\omega}\hat{p}^2 + \frac{\omega}{2}(\omega + 2\sqrt{\alpha\beta})\hat{x}^2$$

For *z=1* one has similarly $\varepsilon = -(\alpha-\beta)/(2(\omega-\alpha-\beta))$

$$\Theta = S^2 = \exp\left(-\frac{\alpha-\beta}{\omega-\alpha-\beta}\omega\hat{x}^2\right)$$

and

$$h_{S(z=1)} = \frac{\omega-\alpha-\beta}{2\omega}\hat{p}^2 + \frac{\omega\Omega^2}{2(\omega-\alpha-\beta)}\hat{x}^2$$

All forms of h_S are of course isospectral, viz

$$\mu\nu = \Omega^2 = \omega^2 - 4\alpha\beta$$

The classical limit of the hermitized Hamiltonian is

$$E_{\mathrm{cl}} = A^2\Omega^2 / 2\mu(z)$$

A is the classical oscillation amplitude

which is explicitly metric dependent, contrary to a recent conjecture from a perturbative calculation (Mostafazadeh) that it should be metric independent.

Berry connection and curvature

Consider $H(q_1, q_2 \ldots.) \equiv H(q)$, generally non-Hermitian

Write $H(q) = S(q)D(q)S^{-1}(q)$, where $D(q)$ is diagonal in chosen basis

S(q) may be singular

Differentiate wrt q's

$$\frac{\partial H(q)}{\partial q_i} - S(q)\frac{\partial D(q)}{\partial q_i}S^{-1}(q) = \left[A_i(q), H(q)\right].$$

with Berry connection

$$A_i(q) = \frac{\partial S(q)}{\partial q_i}S^{-1}(q).$$

generates change in eigenstates

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Consider singularities of *S(q)* – recall that metric is linked to Hermitization (diagonalization):

$$\text{If } S \text{ hermitizes } H, \text{ then } H \text{ is quasi-Hermitian wrt } \Theta = S^{\dagger}S$$

Proceed by considering commutator with *H*

$$\left[\frac{\partial H(q)}{\partial q_i}, H(q)\right] = \left[[A_i(q), H(q)], H(q)\right].$$

Again resort to Moyal construction to solve operator equation for Berry connection *A*

While there is a \`gauge freedom' in *A* , the Berry **phase** should be unique

Invariance of Berry curvature under `gauge' transformation: Λ(q) diagonal

$$A_i(q) \rightarrow A_i^{'}(q) = A_i(q) + S(q)\frac{\partial \Lambda(q)}{\partial q_i}\Lambda^{-1}(q)S^{-1}(q)$$

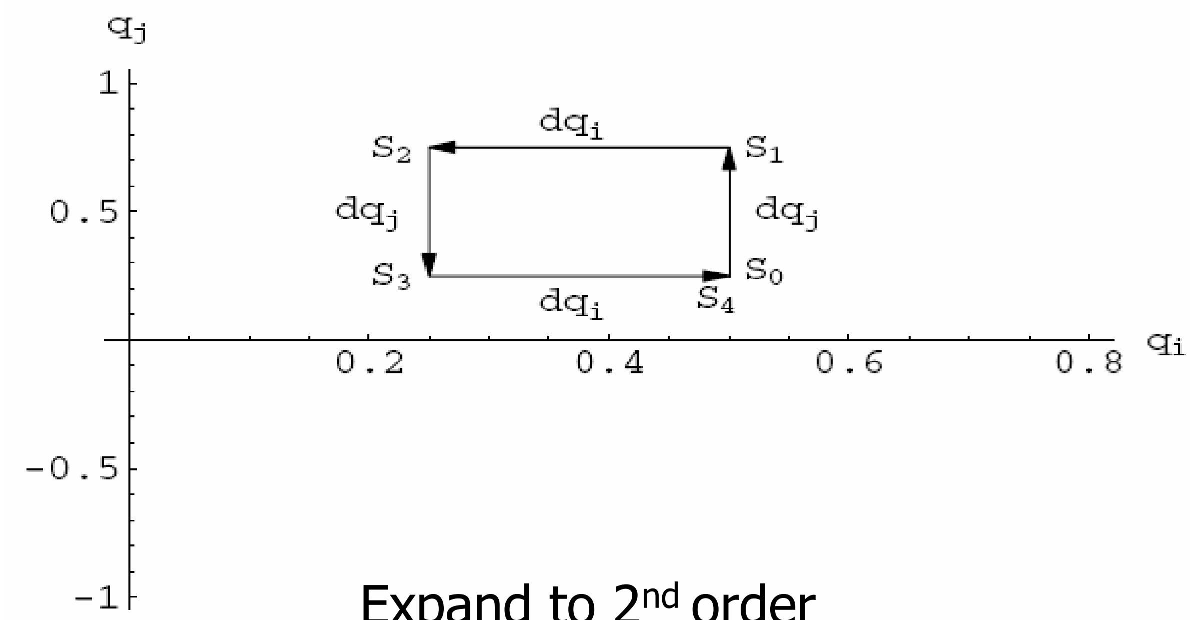
$$S_1 = \left(1 + A_j dq_j + \frac{1}{2}\left(\frac{\partial A_j}{\partial q_j} + A_j{}^2\right)dq_j{}^2\right)S_0 \ ,$$

Change in S to 2nd order around plaquette

$$S_2 = \left(1 + A_i dq_i + \frac{\partial A_i}{\partial q_j} dq_i \ dq_j + \frac{1}{2}\left(\frac{\partial A_i}{\partial q_i} + A_i{}^2\right)dq_i{}^2\right)S_1 \ ,$$

$$S_3 = \left(1 - A_j dq_j - \frac{\partial A_j}{\partial q_i} dq_i \ dq_j + \frac{1}{2}\left(A_j{}^2 - \frac{\partial A_j}{\partial q_j}\right)dq_j{}^2\right)S_2 \ ,$$

$$S_4 = \left(1 - A_i dq_i + \frac{1}{2}\left(A_i{}^2 - \frac{\partial A_i}{\partial q_i}\right)dq_i{}^2\right)S_3 \ .$$



Expand to 2nd order

$$S_1 = \left(1 + A_j dq_j + \frac{1}{2}\left(\frac{\partial A_j}{\partial q_j} + A_j^{\,2} \right) dq_j^{\,2} \right) S_0 \,,$$

$$S_2 = \left(1 + A_i dq_i + \frac{\partial A_i}{\partial q_j} dq_i \, dq_j + \frac{1}{2}\left(\frac{\partial A_i}{\partial q_i} + A_i^{\,2} \right) dq_i^{\,2} \right) S_1 \,,$$

$$S_3 = \left(1 - A_j dq_j - \frac{\partial A_j}{\partial q_i} dq_i \, dq_j + \frac{1}{2}\left(A_j^{\,2} - \frac{\partial A_j}{\partial q_j} \right) dq_j^{\,2} \right) S_2 \,,$$

$$S_4 = \left(1 - A_i dq_i + \frac{1}{2}\left(A_i^{\,2} - \frac{\partial A_i}{\partial q_i} \right) dq_i^{\,2} \right) S_3 \,.$$

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$$\text{yielding } S_4 = \left(1 + F_{ij} dq_i dq_j\right) S_0 \ ,$$

where the Berry curvature has been introduced:

$$F_{ij} = \frac{\partial A_i}{\partial q_j} - \frac{\partial A_j}{\partial q_i} + \left[A_i, A_j\right].$$

Invariance of F under the transformation

$$A_i(q) \rightarrow A_i^{'}(q) = A_i(q) + S(q)\frac{\partial \Lambda(q)}{\partial q_i}\Lambda^{-1}(q) S^{-1}(q) \ ,$$

now readily follows

Solving for the Berry connection from the operator equation

$$\left[\frac{\partial H(q)}{\partial q_i}, H(q)\right] = \left[[A_i(q), H(q)], H(q)\right].$$

Consider a simple 2D matrix model

$$H(q_1,q_2) = \begin{pmatrix} 1 & q_1 + iq_2 \\ q_1 + iq_2 & -1 \end{pmatrix} ;$$

General solution is

$$A_1(q_1,q_2) = \begin{pmatrix} \frac{2\,w_1}{q_1 + i\,q_2} - \frac{1}{(q_1 + i\,q_2)\left(1 + (q_1 + i\,q_2)^2\right)} + y_1 & w_1 - \frac{1}{1 + (q_1 + i\,q_2)^2} \\ w_1 & y_1 \end{pmatrix}$$

$$A_2(q_1,q_2) = \begin{pmatrix} \frac{2\,i\,w_1}{q_1 + i\,q_2} - \frac{i}{(q_1 + i\,q_2)\left(1 + (q_1 + i\,q_2)^2\right)} + y_2 & iw_1 - \frac{i}{1 + (q_1 + i\,q_2)^2} \\ i\,w_1 & y_2 \end{pmatrix}$$

Removing (spurious) singularity at the origin leaves

$$A_1(q_1,q_2) = -i A_2(q_1,q_2) = \begin{pmatrix} \dfrac{q_1 + iq_2}{1 + (q_1 + iq_2)^2} & \dfrac{1}{2} - \dfrac{1}{1 + (q_1 + iq_2)^2} \\ \dfrac{1}{2} & 0 \end{pmatrix}$$

Singularities at $q_1 = 0,\ q_2 = \pm 1$, ie at the exceptional points where H is not diagonalizable

Compute curvature at exceptional point from $\dfrac{\partial S(\phi)}{\partial \phi} S^{-1}(\phi)\Big|_{(0,1)} = A_\phi = \begin{pmatrix} \dfrac{i}{2} & -\dfrac{1}{2} \\ 0 & 0 \end{pmatrix}$.

Result is $F = \begin{pmatrix} -1 & -2i \\ 0 & 1 \end{pmatrix}$.

Eigenstates close to e.p. are $u_\pm = \begin{pmatrix} -i \pm i\sqrt{2w} \\ 0 \end{pmatrix}$

$Fu_\pm = u_\mp$, interchanging eigenstates as anticipated

Analysis for the quadratic boson Hamiltonian

Can express *H as*

$$H(x,p) = p^2 + q_1 x^2 + iq_2 px; \quad q_1 = \frac{b}{a} = \frac{\omega+\alpha+\beta}{\omega-\alpha-\beta},\, q_2 = \frac{c}{a} = \frac{2(\alpha-\beta)}{\omega-\alpha-\beta}.$$

Now solve

$$\left[\frac{\partial H(x,p,q}{\partial q_i}), H(x,p,q)\right]_* = [[A_i(x,p,q), H(x,p,q)], H(x,p,q)]_* .$$

where

$$[A(x,p), B(x,p)]_* = A(x,p) * B(x,p) - B(x,p) * A(x,p) .$$

is the Moyal bracket

Ansatz for A :

$$A_i(x,p) = r_i\, p^2 + s_i\, xp + t_i\, x^2 \; ; i = 1, 2 \, ,$$

yields

$$A_1(x,p) = \frac{i\; p\; x}{\hbar\left(4\, q_1 + {q_2}^2\right)} - \frac{x^2\; q_2}{2\,\hbar\left(4\, q_1 + {q_2}^2\right)} \, ,$$

$$A_2(x,p) = \frac{x^2\; q_2}{\hbar\left(4\, q_1 + {q_2}^2\right)} + \frac{i\; p\; x\, q_2}{2\hbar\left(4\, q_1 + {q_2}^2\right)} \, .$$

Singularities at

$$4q_1 + q_2^2 = \omega^2 - 4\alpha\beta = 0$$

On these curves the Bogoliubov transformation that diagonalizes H breaks down; metric Θ does not exist. Link to quantum phase transition?

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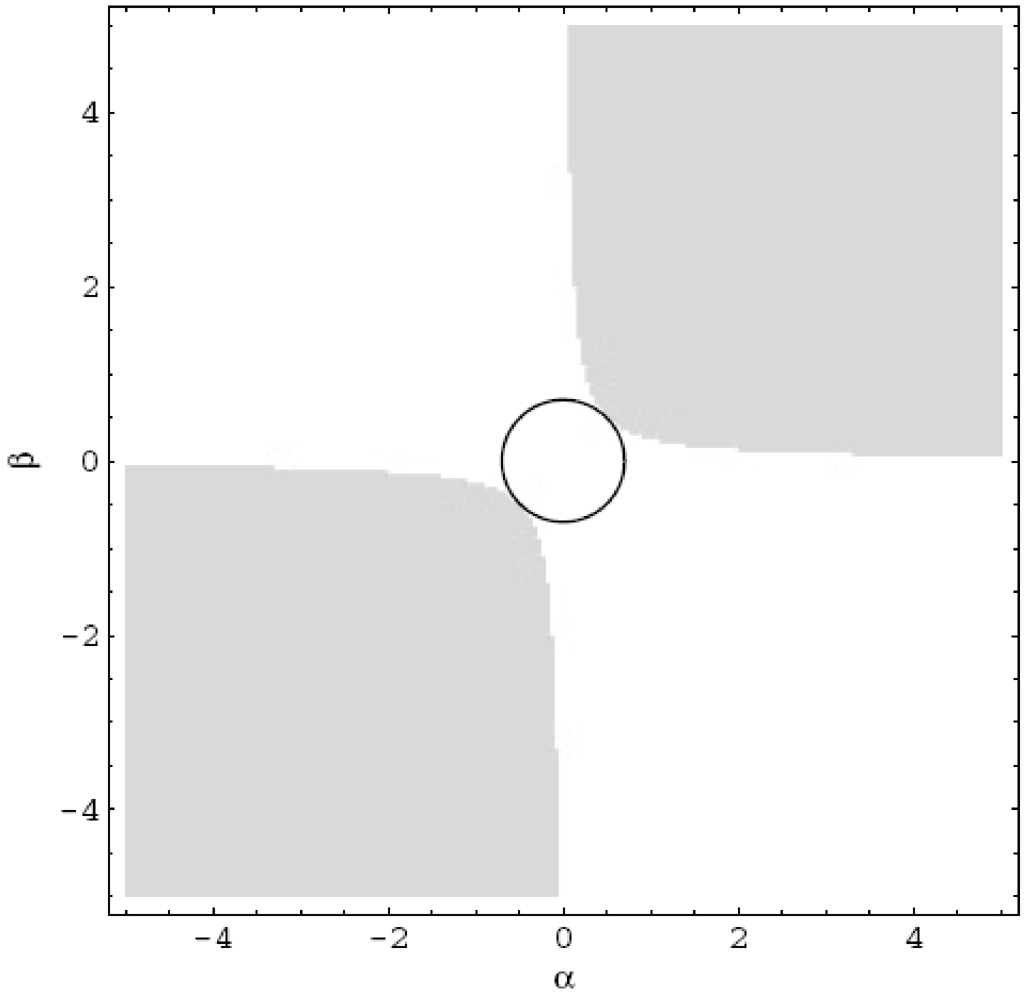

Singular curves of the Berry connection; cannot pass between regions without crossing a singularity of S(q)

Circle is expected radius of convergence for perturbative expansion around the origin.

Conclusions & avenues to explore

- A *consistent* framework of QM can be built on quasi-Hermitian operators; **exists since 1992** and includes $\mathcal{PT}$-symmetric quantum mechanics; central role of metric
- Moyal product construction is a viable route to obtain the metric from its basic operator definition

- Explore non-uniqueness of metric/different choices of irreducible set of observables; choice of observables (and metric) as starting point of QM
- Possible link between phase structure/transitions and singularities of the metric
- Phenomenology of non-Hermitian boson Hamiltonians
- Explicit construction of metric for such models (old problem of CM Vincent & GK Kim)
- Clarify what we understand under a physical application of non-Hermitian QM

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Laboratory for non-Hermitian QM

Experimental PT-symmetric QM

Experimental quasi-Hermitian QM

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