Complex projection of quasianti-Hermitian quaternionic dynamics

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Some recent studies on formulations of quantum mechanics on quaternionic Hilbert spaces have been developed along two seemingly uncorrelated lines.

-On one hand, the complex projection of dynamics generated by (time-independent) quaternionic anti-Hermitian Hamiltonians was considered by showing that they represent one-parameter semigroup dynamics in the space of complex density matrices.

(This can be useful. For instance, in the case of two qubit compound system we showed that the complex projection of quaternionic unitary dynamics between pure states permits the description of interesting phenomena as decoherence and optimal entanglement generation.)

-On the other hand, pseudoanti-Hermitian quaternionic Hamiltonians were introduced in order to generalize standard anti-Hermitian Hamiltonians in quaternionic Hilbert space.

Moreover, we showed that the subclass of complex quasi-Hermitian systems can be described as open quantum systems. A master equation of Lindblad type can be derived for such systems, obtaining one-parameter semigroup dynamics in the space of complex quasi-Hermitian density matrices.

We will show that the complex projection of η-quasianti-Hermitian quaternionic dynamics are one-parameter semigroup dynamics in the space of complex η-quasi-Hermitian density matrices if and only if such dynamics belong to the subclass of η-quasianti-Hermitian ones where η is a (positive definite) complex operator.

Complex projection of QQM

The density matrix ρ_ψ associated with a pure state $|\psi\rangle \in \mathbb{Q}^n$ is defined by

$$\rho_\psi = |\psi\rangle\langle\psi|$$

Mixed states are described by positive quaternionic Hermitian operators $\rho = \rho_\alpha + j\rho_\beta$ on $\mathbb{Q}^n$ with unit trace and rank greater than one.

The expectation value of $A = A_\alpha + jA_\beta$ on a state $|\psi\rangle$ can be expressed as

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle = \mathrm{Re}\,\mathrm{Tr}(A|\psi\rangle\langle\psi|) = \mathrm{Re}\,\mathrm{Tr}(A\rho_\psi).$$

$$\langle A\rangle_\rho = \mathrm{Re}\,\mathrm{Tr}(A\rho) = \mathrm{Re}\,\mathrm{Tr}(A_\alpha\rho_\alpha - A_\beta^*\rho_\beta).$$

Thus, the expectation value of an Hermitian operator A on the state ρ depends on the quaternionic parts of A and ρ, only if both the observable and the state are represented by genuine quaternionic matrices.

This simple observation enables us to merge CQM in the framework of QQM, without modifying any theoretical prediction if only complex observables are taken into account.

Let us denote by $M(\mathrm{Q})$ and $M(\mathrm{C})$ the space of $n \times m$ quaternionic and complex matrices respectively and let $M = M_\alpha + jM_\beta \in M(\mathrm{Q})$. We define the **complex projection**

$$P : M(\mathrm{Q}) \to M(\mathrm{C})$$

by the relation

$$P[M] = \frac{1}{2}[M - iMi] = M_\alpha.$$

Proposition. *The complex projection of a quaternionic density matrix is a complex density matrix.*

When we consider time-independent quaternionic unitary dynamics,

$$\rho(t) = U(t)\rho(0)U^{\dagger}(t),$$

where

$$U(t) = e^{-Ht}$$

with $H = H_{\alpha} + jH_{\beta} = -H^{\dagger}$, the differential equation associated with the time evolution for ρ is given by

$$\frac{d}{dt}\rho(t) = -[H, \rho(t)].$$

The complex projection reads

$$\frac{d}{dt}\rho_{\alpha} = -[H_{\alpha}, \rho_{\alpha}] + H_{\beta}^{*}\rho_{\beta} - \rho_{\beta}^{*}H_{\beta}.$$

It was proven that *the dynamics ruled by the previous equation is a one-parameter semigroup dynamics in the space of complex density matrices* ρ_{α}.

Pseudoanti-Hermitian Q-dynamics

Whenever the quaternionic Hamiltonian H of a quaternionic quantum system is η**-pseudoanti-Hermitian**,

$$\eta H \eta^{-1} = -H^{\dagger}, \quad \eta = \eta^{\dagger},$$

the pseudo-inner product, $(\bullet, \bullet)_{\eta} = (\bullet, \eta \bullet)$, is invariant under the time traslation generated by H.

The adjoint $A^{\ddagger}$ of A with respect to $(\bullet, \bullet)_{\eta}$, is given by

$$A^{\ddagger} = \eta^{-1} A^{\dagger} \eta,$$

so that for any η**-pseudo-Hermitian** operator A,

$$\eta A \eta^{-1} = A^{\dagger},$$

one has,

$$A = A^{\ddagger}.$$

If $A = A^{\ddagger}$, then, $\eta A = (\eta A)^{\dagger}$, so that

$$\langle\psi|\eta A|\psi\rangle = \mathrm{Re}\,\mathrm{Tr}(|\psi\rangle\langle\psi|\eta A) = \mathrm{Re}\,\mathrm{Tr}(\tilde{\rho}A),$$

where $\tilde{\rho} = |\psi\rangle\langle\psi|\eta$.

More generally, if ρ denotes a generic quaternionic density matrix, we can associate it with a **generalized density matrix** $\tilde{\rho}$ by means of

$$\tilde{\rho} = \rho\eta$$

and we obtain $\langle A\rangle_{\eta} = \mathrm{Re}\,\mathrm{Tr}(\tilde{\rho}A)$.

Note that $\tilde{\rho}$ is η-pseudo-Hermitian:

$$\tilde{\rho}^{\dagger} = \eta\rho = \eta\tilde{\rho}\eta^{-1}.$$

Let us consider now the time evolution of a pure state.

If the Hamiltonian H is η-pseudoanti-Hermitian, the evolution operator $V(t) = e^{-Ht}$:

$$|\psi(t)\rangle = V(t)|\psi(0)\rangle$$

is no longer unitary, but η-**Unitary**,

$$V^{\dagger}\eta V = \eta.$$

Hence,

$$\rho(t)\eta = \tilde{\rho}(t) = V(t)\tilde{\rho}(0)V(t)^{-1}.$$

The η-pseudo-norm conservation holds:

$$\mathrm{Re}\,\mathrm{Tr}\tilde{\rho}(t) = \mathrm{Re}\,\mathrm{Tr}\tilde{\rho}(0).$$

The time evolution of $\tilde{\rho}(t)$ is described by the usual Liouville-von Neumann equation:

$$\frac{d}{dt}\tilde{\rho}(t) = -[H, \tilde{\rho}].$$

Complex projection of quasianti-Hermitian Q-dynamics

We will restrict to consider the subclass of η**-quasi-Unitary** dynamics generated by η-pseudoanti-Hermitian quaternionic Hamiltonians H where $\eta = B^\dagger B$.

In this case a new positive definite inner product can be introduced in the Hilbert space where all the usual requirements for a proper quantum mechanical interpretation can be maintained.

For instance, an important property of the corresponding generalized density matrices is that they are positive definite:

$$\rho \geq 0 \Rightarrow B\rho B^\dagger = B\tilde{\rho}B^{-1} \geq 0.$$

The following proposition give us information about the complex projection $\tilde{\rho}_\alpha$ of the η-quasi-Hermitian quaternionic density matrices $\tilde{\rho} = \rho\eta = \tilde{\rho}_\alpha + j\tilde{\rho}_\beta$.

Proposition 1. *The complex projection $\tilde{\rho}_\alpha$ of a η-quasi-Hermitian quaternionic matrix $\tilde{\rho}$ is η-quasi-Hermitian if and only if the entries of η are complex.*

Hence, the subclass of η-quasianti-Hermitian quaternionic dynamics where η is a complex operator allows one to construct η-quasi-Hermitian quaternionic density matrices $\tilde{\rho}$ admitting η-quasi-Hermitian complex projection density matrices $\tilde{\rho}_\alpha$:

$$\rho_\alpha \geq 0 \Rightarrow B\rho_\alpha B^\dagger = B\tilde{\rho}_\alpha B^{-1} \geq 0.$$

Moreover, we can prove that any η-quasianti-Hermitian quaternionic Hamiltonians H with a complex η is given by

$$H = (K + jS)\eta = A\eta,$$

where $K^\dagger = -K$, $S^T = S$, hence $A = -A^\dagger$.

Now, putting $\eta = B^{\dagger}B$ into the previous equation we get

$$H' = BHB^{-1} = -(BHB^{-1})^{\dagger} = -H'^{\dagger},$$

i. e., H' is anti-Hermitian.

In particular, a standard master equation holds for the complex projection of the dynamics generated by H'

$$\frac{d}{dt}\rho'_{\alpha} = L[\rho'_{\alpha}] = -[H'_{\alpha}, \rho'_{\alpha}] + H'^{*}_{\beta}\rho'_{\beta} - \rho'^{*}_{\beta}H'_{\beta}$$

where ρ'_{α} and ρ'_{β} are the complex projection and the purely quaternionic term, of the quaternionic Hermitian density matrix

$$\rho' = \rho'_{\alpha} + j\rho'_{\beta} = B\rho\eta B^{-1} = B\rho B^{\dagger}.$$

The dynamics is a one-parameter semigroup dynamics in the space of complex density matrices, so that we can identify

$$L[\rho'_{\alpha}] = -[H'_{\alpha}, \rho'_{\alpha}] + \sum_{r,s=1}^{n^2-1} C_{rs}(F'_{r}\rho'_{\alpha}F'^{\dagger}_{s} - \frac{1}{2}\{F'^{\dagger}_{r}F'_{s}, \rho'_{\alpha}\}),$$

where $\mathrm{Tr}(F'^{\dagger}_{r}F'_{s}) = \delta_{rs}$ and $[C_{rs}] = [C_{rs}]^{\dagger}$.

Then, coming back by means of

$$B^{-1} \, : \, |\psi'\rangle \to B^{-1}|\psi'\rangle,$$

we obtain an equation of the Lindblad type which describes the most general time evolution of the generalized complex projection density matrix $\tilde{\rho}_\alpha = \rho_\alpha \eta$:

$$\frac{d}{dt}\tilde{\rho}_\alpha = L[\tilde{\rho}_\alpha] = -[H_\alpha, \tilde{\rho}_\alpha] + D[\tilde{\rho}_\alpha]$$

$$- [H_\alpha, \tilde{\rho}_\alpha] + \sum_{r,s=1}^{n^2-1} C_{rs}(F_r\tilde{\rho}_\alpha F_s^\ddagger - \frac{1}{2}\{F_r^\ddagger F_s, \tilde{\rho}_\alpha\})$$

Note that $\mathrm{Tr}(F_r^\ddagger F_s) = \delta_{rs}$ and the dissipative term $D[\tilde{\rho}_\alpha]$ is η-quasi-Hermitian.

Hence, *the dynamics ruled by the previous equation is a one-parameter semigroup dynamics in the space of η-quasi-Hermitian complex density matrices $\tilde{\rho}_\alpha$.*

An Example

We denote by H_α the free complex anti-Hermitian Hamiltonian describing a spin half particle in a constant magnetic field,

$$H_\alpha = \frac{\omega}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and by jH_β a purely η-quasianti-Hermitian quaternionic constant potential,

$$jH_\beta = \begin{pmatrix} 0 & j\frac{v}{x} \\ jxv & 0 \end{pmatrix} \quad (x, v \in \mathrm{R} - \{0\}),$$

where

$$\eta = \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Putting $H = H_\alpha + jH_\beta$, we obtain

$$\eta H \eta^{-1} = -H^\dagger.$$

The eigenvalues and the corresponding biorthonormal eigenbasis of $H = H_\alpha + jH_\beta$ are given by

$$iE_\pm = i\left(\frac{\omega}{2} \pm v\right)$$

and

$$|\psi_\pm\rangle = \begin{pmatrix} \pm\frac{i}{x} \\ j \end{pmatrix} \frac{1}{\sqrt{2}}, \quad |\phi_\pm\rangle = \begin{pmatrix} \pm xi \\ j \end{pmatrix} \frac{1}{\sqrt{2}}.$$

The η-quasi-Unitary evolution operator reads

$$V(t) = e^{-Ht} = |\psi_+\rangle e^{-iE_+t}\langle\phi_+| + |\psi_-\rangle e^{-iE_-t}\langle\phi_-| = \frac{1}{2}\begin{pmatrix} e^{-iE_+t} + e^{-iE_-t} & \frac{1}{x}(e^{-iE_-t} - e^{-iE_+t})k \\ x(e^{iE_+t} - e^{iE_-t})k & e^{iE_+t} + e^{iE_-t} \end{pmatrix}.$$

Let us consider a η-quasi-Hermitian complex pure initial state:

$$\tilde{\rho}(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then,

$$\tilde{\rho}(t) = V(t)\tilde{\rho}(0)V(t)^{-1} = V(t)\tilde{\rho}(0)\eta^{-1}V^{\dagger}(t)\eta =$$

$$\frac{1}{2}\begin{pmatrix} 1-\cos(2vt) & -\frac{j}{x}\sin(2vt) \\ jx\sin(2vt) & 1+\cos(2vt) \end{pmatrix}.$$

The η-quasi-Hermitian complex projection $\tilde{\rho}_{\alpha}(t)$ of $\tilde{\rho}(t)$ assume the diagonal form,

$$\tilde{\rho}_{\alpha}(t) = \frac{1}{2}\begin{pmatrix} 1-\cos(2vt) & 0 \\ 0 & 1+\cos(2vt) \end{pmatrix}.$$

The one-parameter semigroup generator associated with the complex projection of the quaternionic η-quasianti-Hermitian dynamics can be immediately computed:

$$L[\tilde{\rho}_\alpha(t)] = -[H_\alpha, \tilde{\rho}_\alpha] + H^*_\beta \tilde{\rho}_\beta - \tilde{\rho}^*_\beta H_\beta = \begin{pmatrix} v\sin(2vt) & 0 \\ 0 & -v\sin(2vt) \end{pmatrix}.$$

The expectation value of the z-component of the spin observable

$$s_z = \frac{\sigma_3}{2},$$

when the system is in the generalized state $\tilde{\rho}(t)$ is given by (note that s_z is η-quasi-Hermitian)

$$\langle s_z \rangle = \mathrm{Re}\,\mathrm{Tr}(s_z \tilde{\rho}(t)) = \mathrm{Tr}(s_z \tilde{\rho}_\alpha(t)) = \frac{\cos(2vt)}{2}.$$

By a simple calculation the (positive definite) energy η-quasi-Hermitian observable $|H|$ reads

$$|H| = |\psi_+\rangle E_+ \langle \phi_+| + |\psi_-\rangle E_- \langle \phi_-| = \begin{pmatrix} \frac{\omega}{2} & -k\frac{v}{x} \\ kxv & \frac{\omega}{2} \end{pmatrix}$$

and its expectation value is given by

$$\langle |H| \rangle = \mathrm{Re}\,\mathrm{Tr}(|H|\tilde{\rho}(t)) = \frac{\omega}{2} = \mathrm{Tr}(|H_\alpha|\tilde{\rho}_\alpha(t)).$$

This example may have interesting physical applications because the quaternionic potential strongly affects the spin values while the system energy is unchanged.

Concluding remarks

-We proved that the complex projection of η-quasianti-Hermitian quaternionic dynamics are one-parameter semigroup dynamics in the space of complex η-quasi-Hermitian density matrices if and only if such dynamics belong to the subclass of η-quasianti-Hermitian ones where η is a (positive definite) complex operator.

-The complex projection of quaternionic unitary dynamics can be obtained as a particular case of this more general setting, putting $\eta = \mathbf{1}$ into the equations.

-Given the importance of pseudo-Hermitian quantum system dynamics, these results allow us to construct a class of complex η-quasi-Hermitian open quantum system dynamics.

-The inverse problem is under investigation.

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