Pseudo-Hermitian Hamiltonians in Quantum Physics VI.
London, July 16-18, 2007

Projective Hilbert space structures near exceptional points and the quantum brachistochrone problem[1]

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[1] partially based on:
U.G., I. Rotter and B. F. Samsonov, J. Phys. A: Math. Theor. **40**, (2007), 8815-8833, math-ph/0704.1291

Plan of the talk

- Exceptional points (EPs): mathematical background, Jordan blocks
- Motivation
- The parameter space vicinity of EPs
- Projective Hilbert space structures at EPs
- $\mathcal{PT}-$symmetric models
- The quantum brachistochrone problem and the Bloch sphere
- EPs as transformation fixed points
- $\mathcal{PT}-$symmetry, hyperbolic structures and boosted spinors

Exceptional points (EPs): mathematical background, Jordan blocks

- parameter dependent eigenvalue problems:
  parameter space: $\mathcal{M} \ni \mathbf{X} = (X_1, \ldots, X_m), \quad \mathcal{M} \subset \mathbb{C}^m$
  operator: $H(\mathbf{X})$
  eigenvalue problem: $H(\mathbf{X})\Phi(\mathbf{X}) = \lambda(\mathbf{X})\Phi(\mathbf{X})$

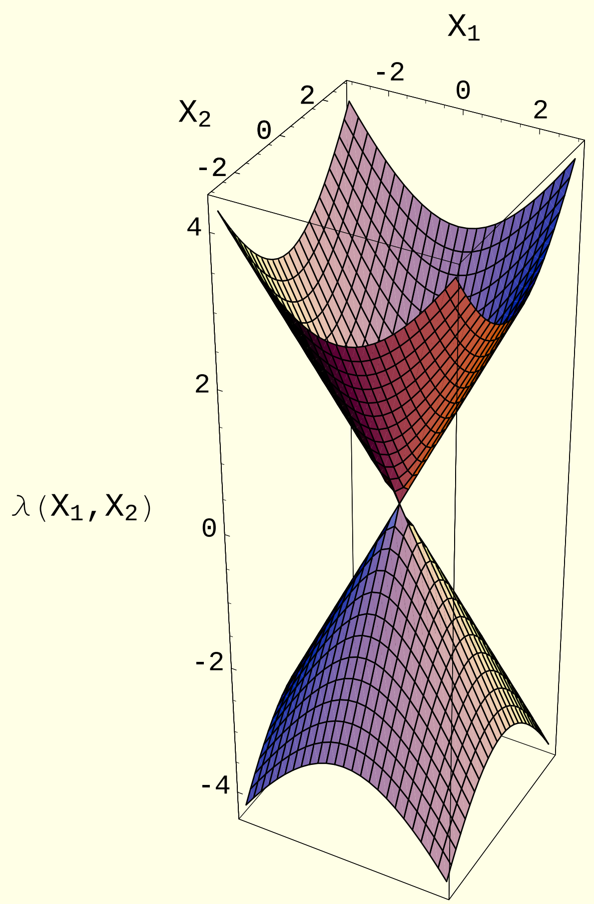
- for simplicity demonstration on matrix eigenvalue problem:
  $H(\mathbf{X}) \in \mathbb{C}^{n \times n}$
  in general n eigenvalues $\lambda_1(\mathbf{X}), \ldots, \lambda_n(\mathbf{X})$,
  i.e. n spectral branches branches over $\mathcal{M} \ni \mathbf{X}$
  with n eigenvectors $\Phi_1(\mathbf{X}), \ldots, \Phi_n(\mathbf{X})$
  diagonalizable:
  $$GHG^{-1} = \operatorname{diag}[\lambda_1, \cdots, \lambda_n]$$

classification of degenerate eigenvalues:

1.) semi-simple eigenvalues:

$$GHG^{-1} = \mathrm{diag}\,[\lambda_0, \cdots, \lambda_0, \lambda_{k+1}, \cdots, \lambda_n]$$

k eigenvalues coalesce
$\lambda_1(\mathbf{X}_d) = \ldots = \lambda_k(\mathbf{X}_d) =: \lambda_0(\mathbf{X}_d)$,
but $\Phi_i(\mathbf{X}_d) \neq \Phi_j(\mathbf{X}_d)$, $\; i, j = 1, \ldots, k$
called diabolical points for $k = 2$
(Michael Berry);
live on hypersurface $\mathcal{V}_d \ni \mathbf{X}_d$, $\; \mathcal{V}_d \subset \mathcal{M}$
enhanced symmetry of the system:
$[A, H(\mathbf{X}_d)] = 0$, $\; A \in U(k)$
rotation in subspace $\mathrm{span}\,[\Phi_1(\mathbf{X}_d), \ldots, \Phi_k(\mathbf{X}_d)]$



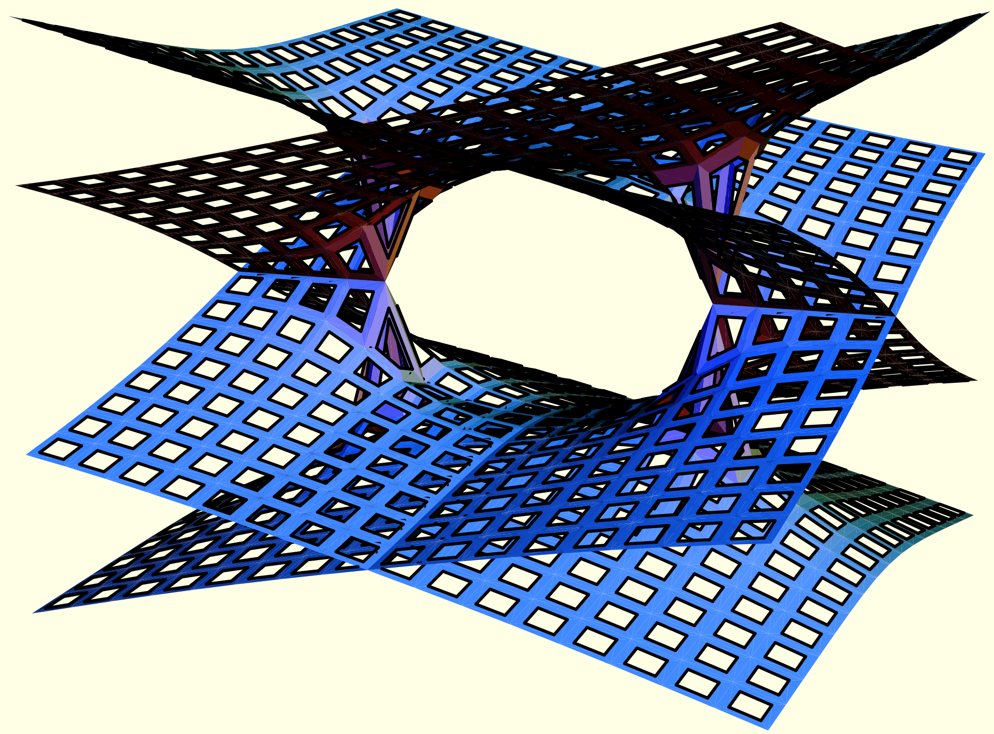
2.) exceptional points (EPs):

$$GHG^{-1} = \mathrm{diag}\left[J_k(\lambda_0), \lambda_{k+1}, \ldots, \lambda_n\right]$$

Jordan block:

$$J_k(\lambda_0) = \begin{pmatrix} \lambda_0 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_0 & 1 \\ 0 & 0 & \cdots & 0 & \lambda_0 \end{pmatrix} \in \mathbb{C}^{k \times k}$$

coalescing spectral branches: $\lambda_j(\mathbf{X} \to \mathbf{X}_c) \to \lambda_0(\mathbf{X}_c)$, j=1,. . . , k
coalescing eigenvectors: $\Phi_j(\mathbf{X} \to \mathbf{X}_c) \to \Theta_0(\mathbf{X}_c)$, j=1,. . . , k
kth-order branch point of the spectral Riemann surface
k spectral branches $\lambda_j(\mathbf{X})$ glued together at the EP



Riemann surface (real component)[2] of $w(z) = \sqrt[4]{(z+1)(z-1)}$
$\exists$ two branch points

[2]Michael Trott, Mathematica© tools for Riemann surfaces, 2000.

- EP-hypersurface $\mathcal{V}_c \ni \mathbf{X}_c,\ \mathcal{V}_0 \subset \mathcal{M}$
- instead of k eigenvectors $\Phi_j(\mathbf{X}),\quad \mathbf{X} \notin \mathcal{V}_c$ there exist k root vectors $\Theta_0(\mathbf{X}_c), \Theta_1(\mathbf{X}_c), \ldots, \Theta_{k-1}(\mathbf{X}_c)$
- root subspace $\mathfrak{S}_\lambda = \mathrm{span}\,[\Theta_0(\mathbf{X}_c), \Theta_1(\mathbf{X}_c), \ldots, \Theta_{k-1}(\mathbf{X}_c)]$

- (geometric) eigenvector $\Theta_0(\mathbf{X}_c)$, algebraic eigenvectors $\Theta_1(\mathbf{X}_c), \ldots, \Theta_{k-1}(\mathbf{X}_c)$
- Jordan chain:

$$\begin{aligned}
[H(\mathbf{X}_c) - \lambda_0 I]\,\Theta_0 &= 0 \\
[H(\mathbf{X}_c) - \lambda_0 I]\,\Theta_1 &= \Theta_0 \\
[H(\mathbf{X}_c) - \lambda_0 I]\,\Theta_2 &= \Theta_1 \\
\cdots \quad &\cdots \quad \cdots \\
[H(\mathbf{X}_c) - \lambda_0 I]\,\Theta_{k-1} &= \Theta_{k-2}
\end{aligned}$$

or
$$[H(\mathbf{X}_c) - \lambda_0 I]^j\,\Theta_{j-1} = 0, \quad j = 1, \ldots, k$$

- far analogy:
  annihilation operator $\hat{a} \approx [H(\mathbf{X}_c) - \lambda_0 I]$
  vacuum state $|0\rangle \approx \Theta_0$
  $j-$particle state $|j\rangle \approx \Theta_j$ with $\hat{a}^j |j\rangle = |0\rangle$
- invariance of the Jordan chain under transformations

$$\Theta_j \mapsto \tilde{\Theta}_j = \Theta_j + a_j \Theta_{j-1}, \quad a_j \in \mathbb{C}$$

- the structure is similar to cohomology chains of differential forms
- multiple Jordan blocks for the same λ
- root subspace: $\mathfrak{S}_\lambda(H) = \bigcup_{n=0}^{\infty} \mathrm{Ker}\left((H - \lambda I)^n\right)$
- geometric multiplicity: $m_\lambda^g(H) = \dim \mathrm{Ker}\left(H - \lambda I\right)$
- algebraic multiplicity: $m_\lambda^a(H) = \dim \mathfrak{S}_\lambda(H)$

Motivation

- at EPs:
  self-orthogonality (isotropy) $\langle \Xi_0 | \Phi_0 \rangle = 0$ of bi-orthogonal basis vectors
  $\Longrightarrow$ subtleties in perturbation techniques

- 2 different perturbation schemes:
  - approaching EPs from diagonalizable configurations, e.g. [E. Narevicius, P. Serra and N. Moiseyev, Europhys. Lett., 2003]

$$H\Phi_0 = E_0\Phi_0, \quad [A, H] \neq 0 \qquad \Longrightarrow \qquad \left| \frac{\langle \Xi_0 | A | \Phi_0 \rangle}{\langle \Xi_0 | \Phi_0 \rangle} \right| \to \infty$$

  - extending the root vector normalization from EPs to their vicinities

$$\langle \Xi_1 | \Phi_0 \rangle = \langle \Xi_0 | \Phi_1 \rangle = 1, \qquad \Longrightarrow \qquad \left| \frac{\langle \Xi_1 | A | \Phi_0 \rangle}{\langle \Xi_1 | \Phi_0 \rangle} \right| < \infty$$

  [A. Sokolov, A. Andrianov and F. Cannata, J. Phys. A, 2006]

$2 \times 2-$**matrix toy model**

- Hamiltonian: $H = \begin{pmatrix} \epsilon_1 & \omega \\ \omega & \epsilon_2 \end{pmatrix}, \quad H = H^T, \quad \omega, \epsilon_{1,2} \in \mathbb{C}$

- convenient parametrization when $\omega \neq 0$:

$$\begin{aligned} H &= E_0 \otimes I_2 + \omega \begin{pmatrix} Z & 1 \\ 1 & -Z \end{pmatrix} \\ E_0 &:= \frac{1}{2}(\epsilon_1 + \epsilon_2), \qquad Z := \frac{\epsilon_1 - \epsilon_2}{2\omega} \\ E_\pm &= E_0 \pm \omega\sqrt{Z^2+1} \\ \Phi_\pm &= \begin{pmatrix} 1 \\ -Z \pm \sqrt{Z^2+1} \end{pmatrix} c_\pm\,; \quad c_\pm \in \mathbb{C}^* := \mathbb{C} - \{0\} \end{aligned}$$

Dual basis and bi-orthogonality

- standard Hilbert space techniques are useless:

$$\begin{aligned} \langle \Phi_+ | \Phi_- \rangle &\equiv \Phi_+^{*T} \Phi_- \\ &= c_+^* c_- \left[1 + |Z|^2 - |Z^2 + 1| + 2\mathrm{Im}\left(Z^* \sqrt{Z^2+1} \right) \right] \\ \langle \Phi_+ | \Phi_- \rangle &= 0 \qquad \Longleftrightarrow \qquad \mathrm{Im} Z = 0 \end{aligned}$$

- dual (left) basis:

$$(H^+ - E_\pm^*)\Xi_\pm = 0, \qquad \langle \Xi_k | \Phi_l \rangle \propto \delta_{kl}, \quad k, l = \pm$$

- complex symmetric matrix $H = H^T$: $\qquad \Longrightarrow \qquad \Xi_\pm \propto \Phi_\pm^*$

- most general ansatz:

$$\Phi_\pm = c_\pm \chi_\pm, \quad \Xi_\pm = d_\pm^* \chi_\pm^*, \quad c_\pm, d_\pm \in \mathbb{C}^*$$

$$\chi_\pm := \begin{pmatrix} 1 \\ -Z \pm \sqrt{Z^2+1} \end{pmatrix}$$

- bi-orthogonality: $\langle \Xi_\pm | \Phi_\mp \rangle = d_\pm c_\mp \chi_\pm^T \chi_\mp = 0 \qquad \forall Z \in \mathbb{C}$

- possible normalization: $\langle \Xi_\pm | \Phi_\pm \rangle = d_\pm c_\pm \chi_\pm^T \chi_\pm = 1$

Projective Hilbert space structures

- Hilbert space: $\Phi_\pm, \Xi_\pm \in \mathcal{H} = \mathbb{C}^2 \approx \mathbb{R}^4$
- line structure due to free scale parameters $c_\pm, d_\pm \in \mathbb{C}^*$
- projective space structure: $\mathbb{P}(\mathcal{H}) = \mathcal{H}^*/\mathbb{C}^* = \mathbb{CP}^1 \ni \pi(\Phi_\pm), \pi(\Xi_\pm)$
- homogeneous coordinates: $\mathbb{CP}^1 \ni (u_0, u_1)$
- topology: $\mathbb{CP}^1 \approx S^3/S^1 \approx S^2$ Riemann sphere
- affine coordinate charts:

$$U_0 \ni (1, u_1/u_0),\ u_0 \neq 0, \qquad U_1 \ni (u_0/u_1, 1),\ u_1 \neq 0$$

$$U_0 \ni (1, z), \quad U_1 \ni (w, 1), \qquad w = 1/z, \qquad w = 0 = 1/\infty$$

- identification: $\chi^T = (1, -Z \pm \sqrt{Z^2+1}) \in U_0 \subset \mathbb{CP}^1$
- natural line bundle: $L = \{(p,v) \in \mathbb{P}(\mathcal{H}) \times \mathcal{H} |\, v = cp,\, c \in \mathbb{C}^*\}$
- $\Phi_\pm, \Xi_\pm$ sections of L: $\Phi_\pm = \pi(\Phi_\pm) \otimes c_\pm$, $\Xi_\pm = \pi(\Xi_\pm) \otimes d^*_\pm$
- locally trivial: $\pi^{-1}(U_0) \approx U_0 \times \mathbb{C}^* \ni \Phi_\pm$

Jordan structure

- setup: $E_\pm = E_0 \pm \omega\sqrt{Z^2+1}\,, \quad \Phi_\pm = \begin{pmatrix} 1 \\ -Z \pm \sqrt{Z^2+1} \end{pmatrix} c_\pm$

- EP-limit: $E_+ = E_- = E_0, \qquad Z^2 = -1, \quad Z = Z_c := \pm i$

- coalescence of lines: $\pi(\Phi_+) = \pi(\Phi_-) =: \pi(\Phi_0)$

$$\chi_+ = \chi_- = \chi_0 := \begin{pmatrix} 1 \\ -Z_c \end{pmatrix} = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$$

- not necessarily coalescence of vectors: $\Phi_+ \neq \Phi_-$

  $\Phi_+ = c_+\chi_0, \qquad \Phi_- = c_-\chi_0$

- bi-orthogonality $\longrightarrow$ isotropy: $\chi_\pm^T \chi_\mp = 0 \quad \longrightarrow \quad \chi_0^T\chi_0 = 0$

- dual Jordan chains:

$$[H(Z_c) - E_0 I]\Phi_0 = 0, \qquad [H(Z_c) - E_0 I]^+ \, \Xi_0 = 0,$$
$$[H(Z_c) - E_0 I]\Phi_1 = \Phi_0, \qquad [H(Z_c) - E_0 I]^+ \, \Xi_1 = \Xi_0$$

- bi-orthogonality:

$$\langle \Xi_0 | \Phi_0 \rangle = \langle \Xi_1 | \Phi_1 \rangle = 0$$
$$\langle \Xi_0 | \Phi_1 \rangle = \langle \Xi_1 | \Phi_0 \rangle = d_0 c_0 \neq 0$$

- explicit representation:

$$\Phi_0 = \sigma q c_0 \begin{pmatrix} 1 \\ -Z_c \end{pmatrix}, \qquad \Phi_1 = \sigma q^{-1} c_0 \begin{pmatrix} -Z_c \\ 1 \end{pmatrix}$$

$$\Xi_0 = \sigma q^* d_0^* \begin{pmatrix} -Z_c \\ 1 \end{pmatrix}, \qquad \Xi_1 = \sigma q^{*-1} d_0^* \begin{pmatrix} 1 \\ -Z_c \end{pmatrix}$$

$$\sigma := \frac{e^{i\mu\frac{\pi}{4}}}{\sqrt{2}},\ q := \sqrt{2\omega}, \qquad Z_c = \pm i =: \mu i, \ \ c_0, d_0 \in \mathbb{C}^*$$

- the whole root subspace $\mathfrak{S}_{E_0}$ scales with the same factor c_0 or d_0^* $\Longrightarrow$ not line structure, but hyperplane structure (beyond usual projective space; projective flag manifold)

- again possible: $\qquad \Phi_{0,a} \neq \Phi_{0,b} \qquad \pi(\Phi_{0,a}) = \pi(\Phi_{0,b}) = \pi(\Phi_0)$

- EP-vicinity: $Z = Z_c + \varepsilon, \qquad |\varepsilon| \ll 1, \qquad \varepsilon \in \mathbb{C}$

- instead of Taylor expansion it holds Puiseux expansion:

$$
\begin{aligned}
E_\pm &= E_0 \pm \varepsilon^{1/2} \Delta E + o(\varepsilon^{1/2}), \\
\Delta E &:= \omega \sqrt{2 Z_c}\,, \\
\chi_\pm &= \begin{pmatrix} 1 \\ -Z_c \end{pmatrix} \pm \varepsilon^{1/2} \begin{pmatrix} 0 \\ \sqrt{2 Z_c} \end{pmatrix} + o(\varepsilon^{1/2})
\end{aligned}
$$

- representation:

$$
\begin{aligned}
\Phi_\pm &= \Phi_0 + \varepsilon^{1/2} (b_0 \Phi_0 + b_1 \Phi_1) + o(\varepsilon^{1/2}) \\
\Xi_\pm^* &= \Xi_0^* + \varepsilon^{1/2} (b_0 \Xi_0^* + b_1 \Xi_1^*) + o(\varepsilon^{1/2}) \\
b_0 &= \pm \frac{Z_c}{2\omega} \Delta E, \qquad b_1 = \pm \Delta E
\end{aligned}
$$

Inner product

- fiber (vector) fitting, $\exists$ two options:
  - primary: root vector scales c_0, d_0
    secondary: $c_+ = c_- = \sigma q c_0, \qquad d_+ = d_- = \sigma^* q Z_c d_0$

    intuitive picture: structure at EP extrapolated into its vicinity

  - primary: scales c_+, c_-, d_+, d_-
    secondary: root vector scales $c_{0,\pm} = c_\pm/(\sigma q),\ \ d_{0,\pm} = d_\pm/(\sigma^* q Z_c)$

    intuitive picture: structure of EP-vicinity extrapolated to EP-limit

- limiting behavior:

$$\begin{aligned}\langle\Xi_\pm|\Phi_\pm\rangle &= 2b_1 d_{0,\pm} c_{0,\pm}\varepsilon^{1/2} + o(\varepsilon^{1/2})\\ &= \frac{2b_1}{\omega Z_c} d_\pm c_\pm \varepsilon^{1/2} + o(\varepsilon^{1/2})\end{aligned}$$

- two different normalization schemes for $\varepsilon \to 0$:
  - root vector normalization:
    $\langle\Xi_0|\Phi_1\rangle = \langle\Xi_1|\Phi_0\rangle = d_0 c_0 \neq 0,\ |d_0 c_0| < \infty$
    $\Longrightarrow \quad \langle\Xi_\pm|\Phi_\pm\rangle \propto \varepsilon^{1/2} \to 0$ (isotropy limit)
  - fixed normalization: $\langle\Xi_\pm|\Phi_\pm\rangle = 1$
    $\Longrightarrow \quad |d_\pm c_\pm| \propto |\varepsilon|^{-1/2} \to \infty$ (scale divergency)
- both normalization schemes are regular for $\varepsilon \neq 0$
- obviously two different charts of a larger unified setup
  $\mathcal{H} \hookrightarrow \mathbb{CP}^2$: $\quad \pi(\Phi_\pm) \times \mathbb{C}^* \hookrightarrow \pi(\Phi_\pm) \times \mathbb{CP}^1$

Simple special case

- setups with $\Xi_m = \Phi_m^*, \qquad c_\pm = d_\pm$
- two normalization schemes:
  - root vector normalization: $\langle \Xi_0 | \Phi_1 \rangle = \langle \Xi_1 | \Phi_0 \rangle = d_0 c_0 = 1$
  - "diagonal" normalization: $\langle \Xi_\pm | \Phi_\pm \rangle = d_\pm c_\pm \chi^T \chi = 1$
- root vector normalization: $\langle \Xi_0 | \Phi_1 \rangle = \langle \Xi_1 | \Phi_0 \rangle = c_0^2 = 1$

  $\Longrightarrow \quad c_0 = \pm 1, \qquad c_\pm = d_\pm = c_0 \sigma q$ rigidly fixed

  $\Longrightarrow$ no geometric phase

- “diagonal” normalization:

$$1 = \langle \Xi_\pm | \Phi_\pm \rangle = \Phi_\pm^T \Phi_\pm \;=\; \left[1 + \left(Z \mp \sqrt{Z^2+1}\right)^2\right] c_\pm^2$$
$$\approx \; \mp 2 Z_c \sqrt{2 Z_c \varepsilon}\, c_\pm^2$$

- scale factors: $c_\pm^2 \approx \mp 2^{-3/2} Z_c^{-3/2} \varepsilon^{-1/2} \quad \Longrightarrow \quad c_\pm \sim \varepsilon^{-1/4}$

  4-fold winding; correct geometric phase

- divergent vector norm: $||\Phi_\pm||^2 = \langle \Phi_\pm | \Phi_\pm \rangle \approx 2|c_\pm|^2 \approx |2\varepsilon|^{-1/2} \to \infty$

- projective space resolution of the singularity:

  $\Phi \in \mathcal{H} \approx \mathbb{C}^2 \hookrightarrow \mathbb{CP}^2 \ni \phi = (u_0, u_1, u_2)$

- embedding trick:

  $\Phi^T = c(1, w) = (z_0, z_1) \hookrightarrow (z_0, z_1, 1) = (\frac{u_0}{u_2}, \frac{u_1}{u_2}, 1) \in U_2 \subset \mathbb{CP}^2$

  $\Longrightarrow \qquad u_2 = c^{-1} \qquad \Longrightarrow \qquad \phi = (1, w, c^{-1}) \in \mathbb{CP}^2$

- resolution of the singularity: $\quad |c| \to \infty \quad \Longrightarrow \quad u_2 \to 0$ beyond U_2

  affine chart $U_0 \ni (1, \frac{u_1}{u_0}, \frac{u_2}{u_0})$ is most convenient:

  $\Phi \approx (1, w, c^{-1}) = (\chi^T, c^{-1}) \approx (\pi(\Phi), c^{-1})$

- normalization condition as constraint:

$$\begin{aligned}
\Phi^T\Phi - 1 &= 0 \\
\frac{u_0^2}{u_2^2} + \frac{u_1^2}{u_2^2} - 1 &= 0 \\
u_0^2 + u_1^2 - u_2^2 &= 0 \\
\chi^T\chi - c^{-2} &= 0
\end{aligned}$$

  conic (singular quadric) in homogeneous coordinates

- extends straight forwardly to higher dimensions: $\mathcal{H} = \mathbb{C}^n \hookrightarrow \mathbb{CP}^n$

  $\sum_{k=0}^{n-1} u_k^2 - u_n^2 = 0$

$\mathcal{PT}-$symmetric models

- $\mathcal{PT}-$symmetry: $\quad [\mathcal{PT}, H] = 0, \qquad \mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \qquad r, s, \theta \in \mathbb{R}$$

- eigenvalues: $\quad E_{\pm} = r\cos(\theta) \pm \sqrt{s^2 - r^2\sin^2(\theta)}$
- eigenvectors:

$$|E_+\rangle = \frac{e^{i\alpha/2}}{\sqrt{2\cos(\alpha)}} \begin{pmatrix} 1 \\ e^{-i\alpha} \end{pmatrix} =: c_+\chi_+ \,, \qquad \sin(\alpha) = \frac{r}{s}\sin(\theta)$$

$$|E_-\rangle = \frac{ie^{-i\alpha/2}}{\sqrt{2\cos(\alpha)}} \begin{pmatrix} 1 \\ -e^{i\alpha} \end{pmatrix} =: c_-\chi_-$$

- dynamical operator $\mathcal{C}$: $[\mathcal{C}, H] = 0$, $\mathcal{C} = \frac{1}{\cos(\alpha)} \begin{pmatrix} i\sin(\alpha) & 1 \\ 1 & -i\sin(\alpha) \end{pmatrix}$

- inner products:

$$\text{Krein space: } (u, v) = \mathcal{PT}u \cdot v : \qquad (E_\pm, E_\pm) = \pm 1, \quad (E_\pm, E_\mp) = 0$$

$$\text{Hilbert space: } ((u, v)) = \mathcal{CPT}u \cdot v : \qquad ((E_\pm, E_\pm)) = 1, \quad ((E_\pm, E_\mp)) = 0$$

- EP-related parametrization: $Z = i\frac{r}{s}\sin(\theta) = i\sin(\alpha)$,
$\mathcal{C} = \frac{1}{\cos(\alpha)} \begin{pmatrix} Z & 1 \\ 1 & -Z \end{pmatrix}$

- Hamiltonian: $H = E_0 I_2 + s\cos(\alpha)\mathcal{C}$, $[\mathcal{C}, H] = 0$ trivially fulfilled

- eigenvectors: $\Phi = c(1,b)^T$

- exact $\mathcal{PT}-$symmetry (PTS):
  $\mathcal{PT}\Phi = c^*(b^*,1)^T = c^*b^*(1,1/b^*)^T \propto \Phi = c(1,b)^T \quad \Longrightarrow \quad |b| = 1$

- compatibility: $\mathcal{PT}\Phi \propto \Xi^*$, $\mathcal{CPT}\Phi \propto \Xi^*$

- orthogonality: $\mathcal{CPT}\Phi_k \cdot \Phi_l \propto \mathcal{PT}\Phi_k \cdot \Phi_l \propto \Xi_k^+\Phi_l$

- energy:

$$
\begin{aligned}
E_\pm &= r\cos(\theta) \pm s\sqrt{1 - \frac{r^2}{s^2}\sin^2(\theta)} \\
&= r\cos(\theta) \pm s\sqrt{1 - \sin^2(\alpha)}
\end{aligned}
$$

- exact PTS: $\alpha \in \mathbb{R} - \{\pi/2 + \pi\mathbb{Z}\}$, $Z \in (-i, i)$, $\mathrm{Re}Z = 0$
- Hermitian Hamiltonian: $\alpha = n\pi$, $n \in \mathbb{Z}$, $Z = 0$
- spontaneously broken PTS:
  $\alpha \in \pi(1/2 + \mathbb{Z}) + i\mathbb{R}$, $Z \in (-i\infty, -i) \cup (i, i\infty)$
- EPs: $Z_c = \pm i$, $\alpha_c = \pi(1/2 + N)$, $N \in \mathbb{Z}$
- line coalescence at EPs: $\pi(|E_+\rangle) = \pi(|E_-\rangle) \approx \chi_0 = (1, Z_c)^T$
- diverging norm: $|||E_\pm\rangle||^2 = \langle E_\pm|E_\pm\rangle \approx \frac{1}{|\cos(\alpha)|} \to \infty$
- Krein-space $\rightleftarrows$ Hilbert space mapping singularity:

$$\mathcal{C} = \frac{1}{\cos(\alpha)} \begin{pmatrix} Z & 1 \\ 1 & -Z \end{pmatrix}, \qquad \cos(\alpha \to \pi/2) \to 0$$

- $\mathcal{PT}-$symmetric projective structures:
  affine coordinate embedding

$$|E_\pm\rangle^T = c_\pm(1, b_\pm) \hookrightarrow (c_\pm, c_\pm b_\pm, 1) \in U_2 \subset \mathbb{CP}^2$$

  homogeneous coordinates: $e_\pm = (1, b_\pm, c_\pm^{-1}) \in \mathbb{CP}^2$

- normalization: $\mathcal{PT}|E_\pm\rangle \cdot |E_\pm\rangle = 1$

- generalized conic: $\mathcal{PT}\chi_\pm \cdot \chi_\pm - \left(\mathcal{T}c_\pm^{-1}\right) c_\pm^{-1} = 0$
  regular in the EP-limit $\alpha \to \alpha_c$

Brachistochrone problem:

- Hermitian Hamiltonians:
  [D.C. Brody, J. Phys. A 2003]
  [A. Carlini, A. Hosoya, T. Koike, Y. Okudaira, Phys. Rev. Lett. 2006]

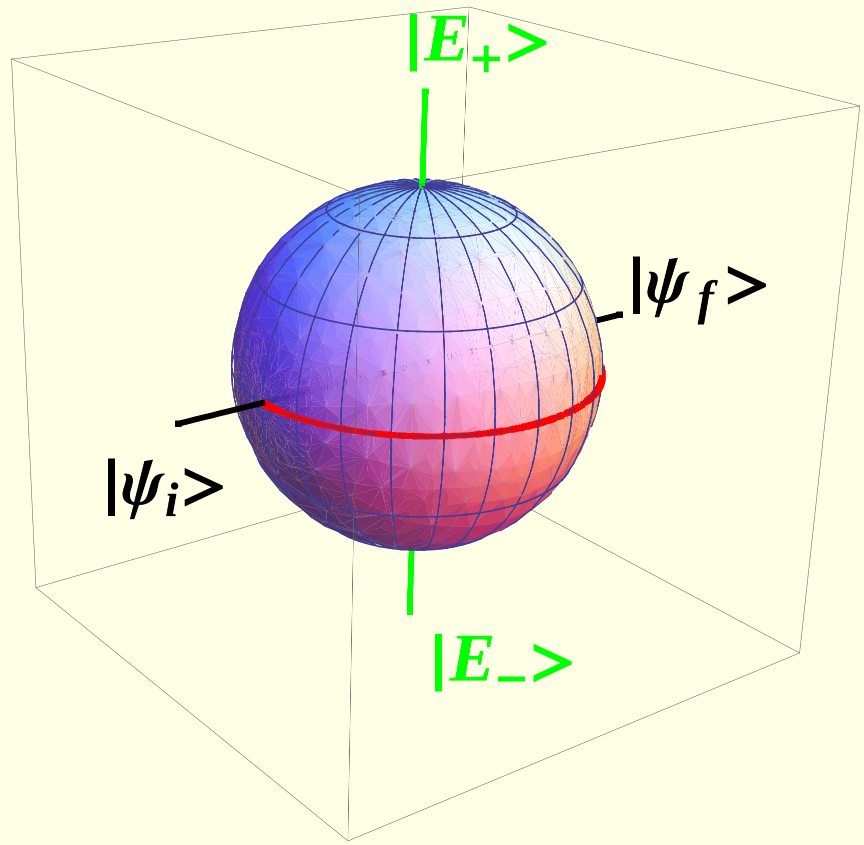
- $\mathcal{PT}-$symmetric Hamiltonians:
  [C.M. Bender, D.C. Brody, H.F. Jones, B.K. Meister, Phys. Rev. Lett., 2007]
  [A. Mostafazadeh, quant-ph/0706.3844]

- general non-Hermitian Hamiltonians:
  [P. E. G. Assis and A. Fring, quant-ph/0703254]

Bloch sphere:

$$|\psi\rangle = \cos(\theta)|0\rangle + e^{i\phi}\sin(\theta)|1\rangle, \qquad \theta \in [0, \pi/2), \quad \phi \in [0, 2\pi)$$

$$\begin{aligned} x &= \sin(2\theta)\cos(\phi) \\ y &= \sin(2\theta)\sin(\phi) \\ z &= \cos(2\theta) \end{aligned}$$



shortest path: geodesic distance between antipodal (orthogonal) points: $\Delta = \pi$

$e^{-itH}|E_\pm\rangle = e^{-itE_\pm}|E_\pm\rangle$

$|E_\pm\rangle$ fixed points

Hermitian system

Non-Hermitian system:

exact $\mathcal{PT}-$symmetry

$H|E_\pm\rangle = E_\pm|E_\pm\rangle$

$\langle E_+|E_-\rangle \neq 0$

$\mathcal{CPT}E_j \cdot E_k = \langle E_j|\eta|E_k\rangle = \delta_{jk}$

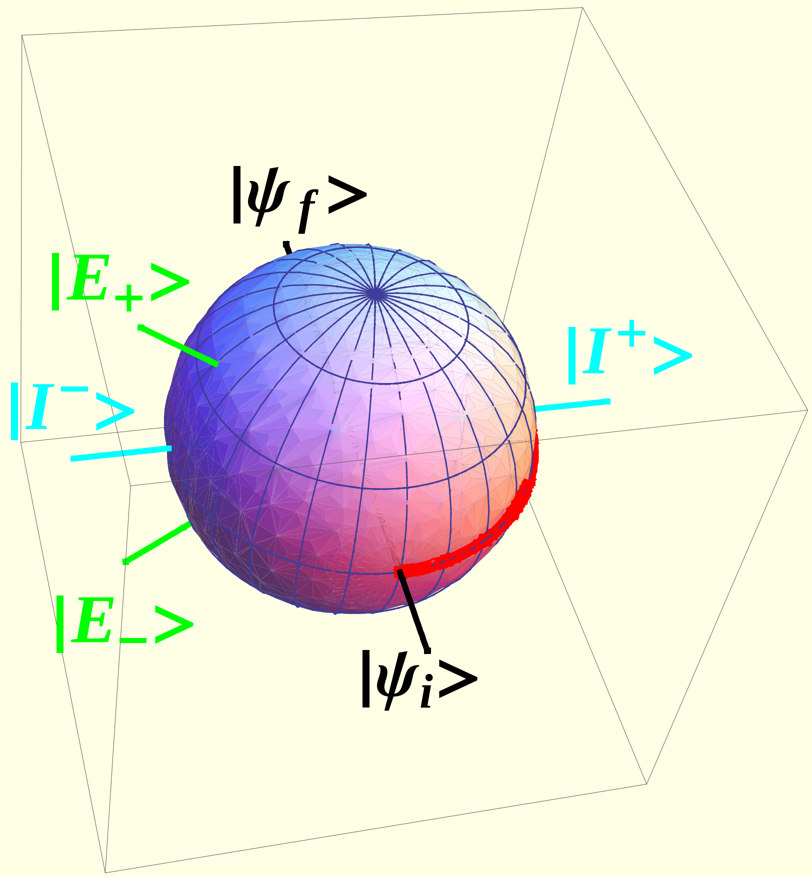
brachistochrone problem: $\{H, |\psi_i\rangle, |\psi_f\rangle\}$

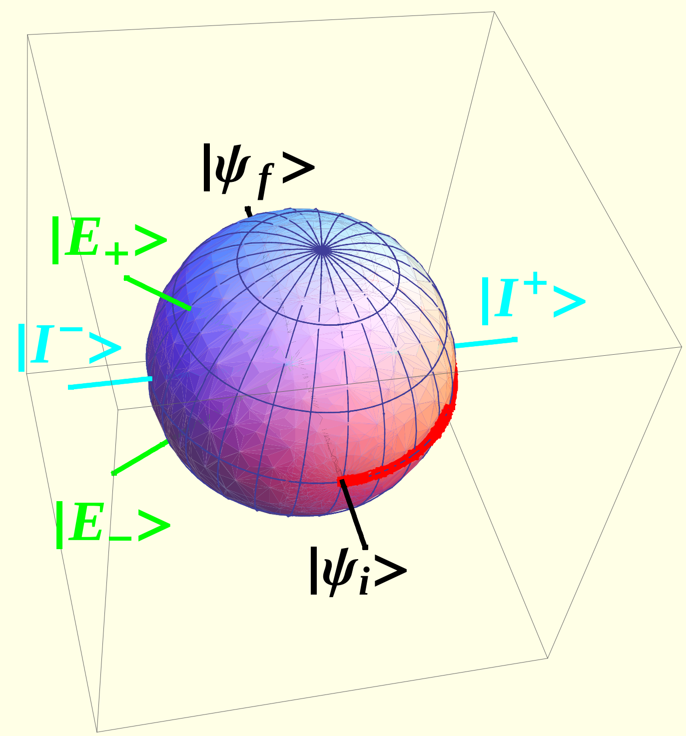
Ali Mostafazadeh:

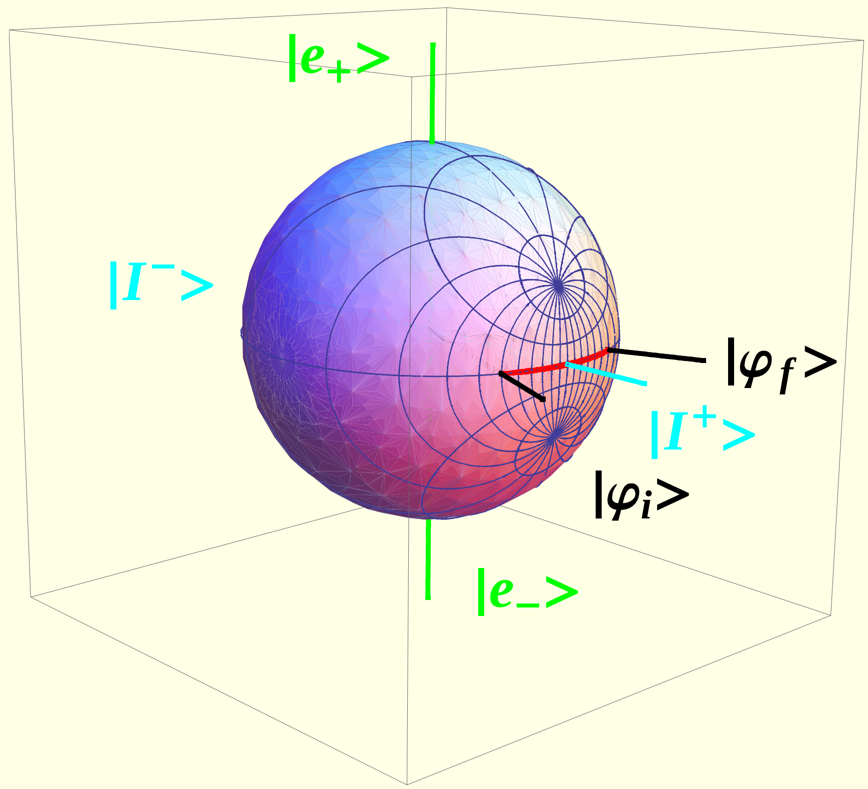
$\eta = \eta^+ > 0$

$$\begin{aligned} \eta^{1/2}: \ H &\mapsto h = \eta^{1/2}H\eta^{-1/2} \\ |\psi_i\rangle &\mapsto |\phi_i\rangle = \eta^{1/2}|\psi_i\rangle \\ |\psi_f\rangle &\mapsto |\phi_f\rangle = \eta^{1/2}|\psi_f\rangle \end{aligned}$$

Hermitian brachistochrone problem $\{h, |\phi_i\rangle, |\phi_f\rangle\}$

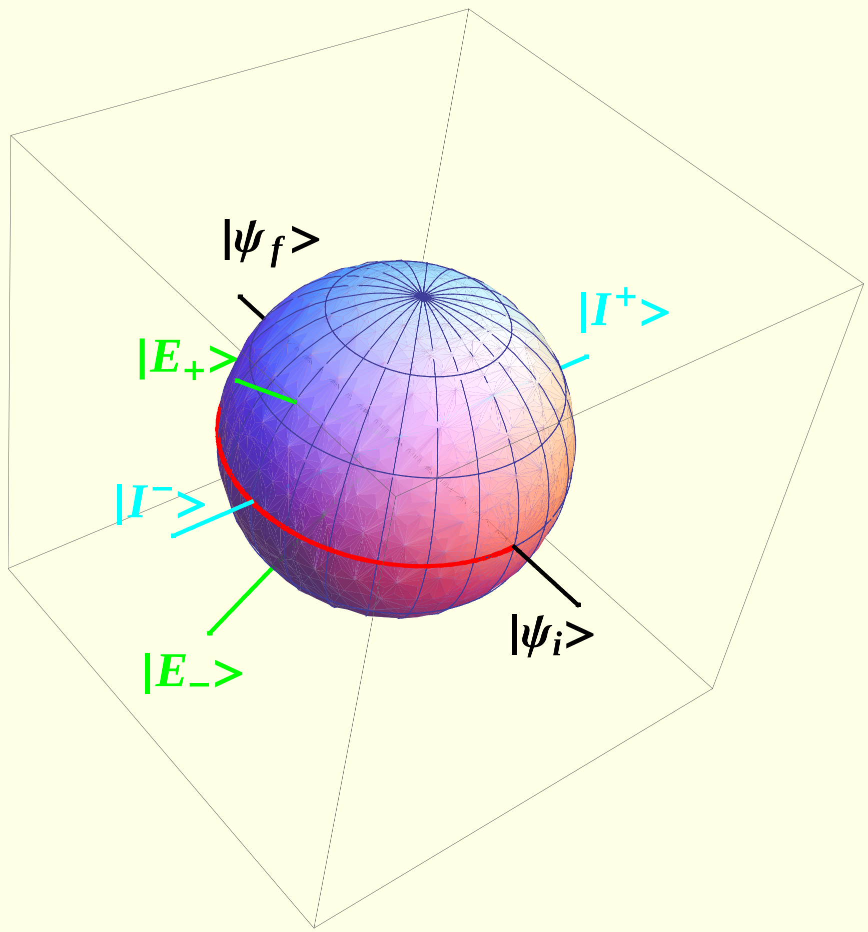




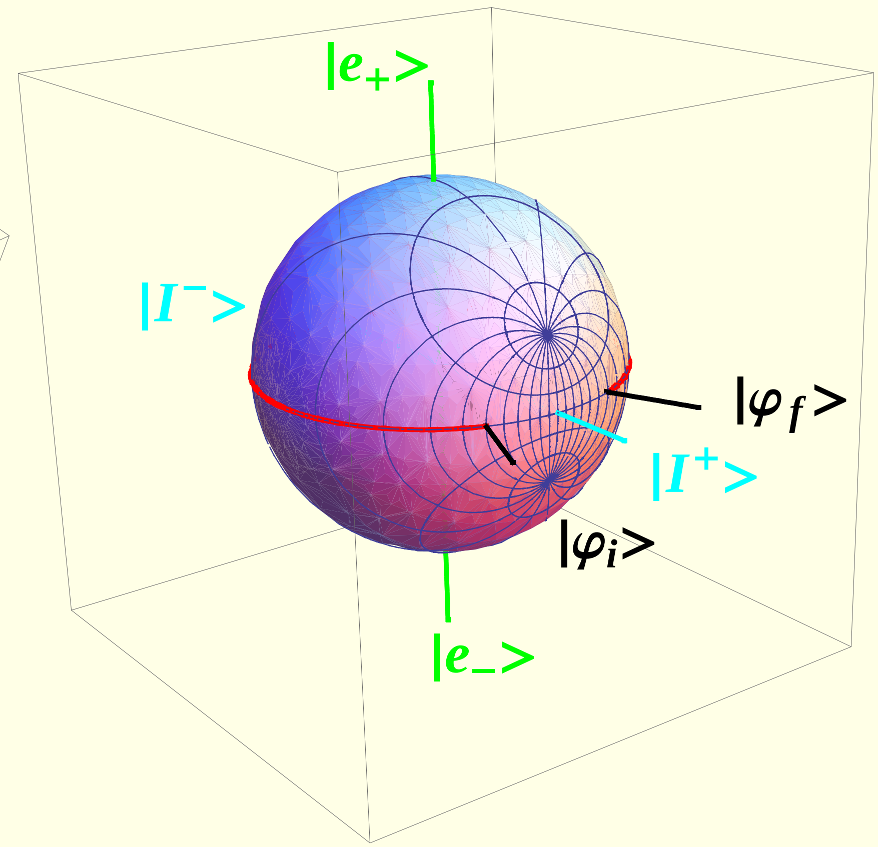


Non-Hermitian problem
$\{H, |\psi_i\rangle, |\psi_f\rangle\}$

equivalent Hermitian problem
$\{h, |\phi_i\rangle, |\phi_f\rangle\}$
evolution path contraction



Non-Hermitian problem
$\{H, |\psi_i\rangle, |\psi_f\rangle\}$



equivalent Hermitian problem
$\{h, |\phi_i\rangle, |\phi_f\rangle\}$
evolution path dilation

EPs as transformation fixed points

- transformation in $\mathbb{C}^2$: $S := \eta^{1/2} \in SL(2,\mathbb{C}) : \; |\psi\rangle \mapsto |\phi\rangle = S|\psi\rangle$
  general form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

- reinterpretation as $PSL(2,\mathbb{C})$ transformation over $\mathbb{CP}^1 \cong \mathbb{C} \cup \infty =: \hat{\mathbb{C}}$

- $PSL(2,\mathbb{C})$ over the affine chart: $(1,z) \mapsto (1,z') = (1,f(z))$

- Möbius transformation: $z' = f(z) = \frac{Dz+C}{Bz+A}$
  $f(z) \in \mathrm{Aut}\,(\hat{\mathbb{C}})$ leads to reparametrization of the Fubini-Study metric on the Bloch sphere

- classification for S with $\det(S) = 1$:
  $(\mathrm{tr}S)^2 = 4$ parabolic
  $(\mathrm{tr}S)^2 \in [0, 4)$ elliptic
  $(\mathrm{tr}S)^2 \in (4, \infty)$ hyperbolic
  $(\mathrm{tr}S)^2 \in \mathbb{C},\ (\mathrm{tr}S)^2 \notin [0, 4]$ loxodromic

- unitary $SU(2)$ transformations: elliptic

- $S = \eta^{1/2}$: $(\mathrm{tr}S)^2 = 2 + \frac{2}{|\cos(\alpha)|}$ hyperbolic

- fixed points: $f_S(z) = z \quad \Longrightarrow \quad z = \pm i \quad \forall \alpha \neq 2\pi N$

- $z = \pm i$ correspond to EP-related isotropic eigenvectors

- all non-Hermitian 2×2 matrix models with exact $\mathcal{PT}-$symmetry have the same two EP-related eigenvectors $|I^{\pm}\rangle$ as transformation fixed points

$\mathcal{PT}-$symmetry, hyperbolic structures and boosted spinors

- qualified guess: EP-related isotropic eigenvectors correspond to "light-cone configurations"

$$
\begin{aligned}
\chi_\pm &= \begin{pmatrix} 1 \\ -Z \pm \sqrt{1+Z^2} \end{pmatrix} = \begin{pmatrix} 1 \\ -i\sin(\alpha) \pm \sqrt{1-\sin^2(\alpha)} \end{pmatrix} \\
&\stackrel{?}{=} \begin{pmatrix} 1 \\ -i\frac{v}{c} \pm \sqrt{1-\frac{v^2}{c^2}} \end{pmatrix} \quad \stackrel{v\to c}{\longrightarrow} \quad \begin{pmatrix} 1 \\ -i \end{pmatrix} = \chi_0
\end{aligned}
$$

- parametrization: $\sin(\alpha) = \tanh(\beta)$

$$\eta = \frac{1}{\cos(\alpha)} \begin{pmatrix} 1 & -i\sin(\alpha) \\ i\sin(\alpha) & 1 \end{pmatrix} = \begin{pmatrix} \cosh(\beta) & -i\sinh(\beta) \\ i\sinh(\beta) & \cosh(\beta) \end{pmatrix}$$

$$= e^{\beta\sigma_y} \in SO(2,\mathbb{C}) \subset SL(2,\mathbb{C}), \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- $S = \eta^{1/2} = e^{\beta\sigma_y/2}$ usual boost acting on spinors

- $\mathcal{PT}-$symmetric brachistochrone problem with fixed $\omega = E_+ - E_-$
- constraint $\omega = \mathrm{const}$ allows for parametrization

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} = r\cos(\theta)I_2 + \frac{\omega}{2}\begin{pmatrix} i\sinh(\beta) & \cosh(\beta) \\ \cosh(\beta) & -i\sinh(\beta) \end{pmatrix}$$

$$h = r\cos(\theta)I_2 + \frac{\omega}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- "mass shell condition" $s^2 - r^2\sin^2(\theta) \equiv \frac{\omega^2}{4}\left[\cosh^2(\beta) - \sinh^2(\beta)\right] = \frac{\omega^2}{4}$

- PTSQM brachistochrone: $\alpha \to \pi/2$ implies $\beta \to \infty$ ultra-relativistic (light-cone) limit

- analogy:
  — h acts as "co-moving" Hamiltonian
  — $H = \eta^{-1/2} h \eta^{1/2}$ describes a process for a rest-frame observer

- problems/obstructions concerning such an interpretation:
  - QM spin system extended to relativistic regimes:
    seems to require interpretation as one component of a two-component (Dirac) spinor (1st system extension)
  - ultra-relativistic limit:
    single particle interpretation of the Dirac spinor is likely to break down due to possible pair-creation processes above threshold energies
  - full QFT approach seems required (2nd system extension):
    the EP-limit might not be reached due to PTSQM system break-down

- other interpretation schemes?