# Lie algebraic approach to non-Hermitian Hamiltonians with real spectra

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**Abstract**

An algebraic technique useful in studying of a non-Hermitian Hamiltonians with real spectra, is presented. The method is illustrated by explicit application to a family of one-dimensional potentials

The existence of non-Hermitian Hamiltonians with real spectra is one of the interesting problems in theoretical physics. For one thing they are used in various branches of theoretical physics, for other it is interesting in itself to understand the reasons for the reality ( see, e.g., [1] and references therein).

The understanding of these Hamiltonians has been largely improved during the past years by the realization that their existence is deeply related to the existence of symmetry under the combined transformation of parity $P$ and time reversal $T$ [2]. Later it was shown that [3] the operator $H$ (with a complete set of biorthonormal eigenvectors) has a real spectrum if there exists a Hermitian automorphism $\eta$ such that

$$H^{\dagger}\eta = \eta H \tag{1}$$

or

$$HO = OH_0 \tag{2}$$

where $OO^{\dagger} = \eta$ and $H_0$ is Hermitian. In a recent paper, however, Kretschmer and Szymanowski proposed a way which might allow for finding in a systematic way large classes of non-Hermitian Hamiltonians with real spectra. The existence of an operator $\Omega$ that intertwine a given non-Hermitian Hamiltonian $H$ and Hermitian one $h$ ensures the reality of the spectrum of $H$

$$H\Omega = \Omega h \tag{3}$$

Here we shall use a group-theoretical methods to construct a class of non-Hermitian operators $H$ with real spectra for which the relation (3) holds. To gain a better understanding of our approach, we illustrate it for Hamiltonians related to $SO(2.1)$. To this end, a few facts from the representation theory of the $SO(2.1)$ are useful [5].

Let $R^{2,1}$ be a three-dimensional pseudo-Euclidean space with bilinear form

$$[\xi,\zeta]=\xi_0\zeta_0-\xi_1\zeta_1-\xi_2\zeta_2. \tag{4}$$

By $SO(2,1)$ we denote the connected component of the group of linear transformations of $R^{2,1}$preserving the form (4). We consider $SO(2,1)$ as acting on $R^{2,1}$on the right.

The principal non-unitary series of representations $T_\sigma$ of the group $SO(2,1)$ are labelled by complex number $\sigma$. They can be realized in the Hilbert space $L^2(S)$ with inner product

$$(f_1,f_2)=\frac{1}{2\pi}\int_S f_1(n)\, f_2^*(n)dn \tag{5}$$

where $S=\{n=(1,\cos\varphi,\sin\varphi)\}$ denotes the circle of radius 1 and $dn=d\varphi$. The representation $T_\sigma$ is defined by

$$T_\sigma(g)f(n)=|(ng)_0|^\sigma\, f\left(\frac{ng}{(ng)_0}\right). \tag{6}$$

The infinitesimal operators $a_0,a_1,a_2$ of the representation $T_\sigma$, corresponding to the one-parameter subgroups $g_0(t),g_1(t)$ and $g_2(t)$ , where $g_0(t)$ is the rotations in the 1-2 plane, while $g_1(t)$ and $g_2(t)$ are the pure Lorentz transformations along the 1 and 2 axes, respectively are given by

$$\begin{aligned} a_1 &= \sigma\cos\varphi-\sin\varphi\frac{d}{d\varphi} \\ a_2 &= -\sigma\sin\varphi-\cos\varphi\frac{d}{d\varphi} \\ a_3 &= \frac{d}{d\varphi} \end{aligned} \tag{7}$$

The Casimir operator

$$C=a_0^2-a_1^2-a_2^2 \tag{8}$$

is identically a multiple of the unit

$$C=-\sigma(\sigma+1)I. \tag{9}$$

The representations $T_\sigma$ and $T_{-\sigma^*-1}$ are Hermitian-adjoint, i.e.

$$(T_\sigma f_1,T_{-\sigma^*-1}f_2)=(f_1,f_2) \tag{10}$$

Therefore $T_\sigma$ is unitary if and only if $\mathrm{Re}\,\sigma=-\frac{1}{2}$. The infinitesimal operators of a unitary representation satisfy the condition

$$a_\alpha^+=-a_i,\quad i=0,1,2 \tag{11}$$

i.e the operators

$$J_k=-ia_k,\quad k=0,1,2 \tag{12}$$

are Hermitian. For $\mathrm{Re}\,\sigma \neq -\frac{1}{2}$ the representation $T_\sigma$ is non-unitary although $J_3$ still Hermitian. If we diagonalize $J_3$ we obtain

$$J_3\psi_m = m\psi_m, \quad C\psi_m = -\sigma(\sigma+1)\psi_m, \quad m = 0, \pm 1, \pm 2, \dots \tag{13}$$

A key concept in group-theoretical approach is that the Hamiltonian $H$ under study is a function of infinitesimal operators $a_i$ of the representation of some Lie group $G$

$$H = \Phi(a_i). \tag{14}$$

Particularly

$$H = \Phi(C_i) \tag{15}$$

where $C_i$ are Casimir operators of $G$. Here we want to construct the Hamiltonians in terms of operators of Lie algebra of $SO(2,1)$ for which relation (3) holds. The key to their construction lies in the observation that the relation (3) for such systems is essentially a relation between equivalent representations of $SO(2,1)$. Thus in order to find the Hamiltonians for the systems under consideration we should look for another realization of principal non-unitary series representation.

Let us denote by $\mathcal{H}^\sigma$ the space of functions $F(\xi)$ on one sheet hyperboloid

$$\xi_0^2 - \xi_1^2 - \xi_2^2 = -1, \tag{16}$$

satisfying the equation

$$\triangle F(\xi) = -\sigma(\sigma+1)F(\xi), \quad \sigma \in \mathbb{C} \tag{17}$$

where

$$\triangle = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} - \wedge(\wedge + 1) \tag{18}$$

with

$$\wedge = \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} \tag{19}$$

Then the principal non-unitary representations of the $SO(2,1)$ can be realized in $\mathcal{H}^\sigma$. In this realization the representation is defined by

$$U_\sigma F(\xi) = F(\xi g) \tag{20}$$

We note that the interrelation between representations (5) and (20) is given by

$$\begin{aligned} F(\xi) &= \int_S |[\xi, n]|^{-1-\sigma} f(n) dn \\ &\equiv (Af)(\xi) \end{aligned} \tag{21}$$

Moreover the following intertwining relation is held

$$U_\sigma A = A T_\sigma \tag{22}$$

We are now prepared to extract the one-dimensional Hamiltonian from (18). For this purpose instead of coordinates $\xi_1$ and $\xi_2$ we introduce the coordinates $x$ and $\theta$ via

$$\xi_1 = \frac{\cos\varphi}{\sqrt{1-z(x)^2}}, \quad \xi_2 = \frac{\sin\varphi}{\sqrt{1-z(x)^2}}, \quad z(x) \in [-1,1] \tag{23}$$

If we compute $\triangle$ for this parametrization it becomes

$$\triangle = \frac{(1-z^2)^2}{\dot{z}^2}\left[-\frac{\partial^2}{\partial x^2} + \left(\frac{z\dot{z}}{1-z^2} + \frac{\ddot{z}}{\dot{z}}\right)\frac{\partial}{\partial x} + \frac{\dot{z}^2}{1-z^2}\frac{\partial^2}{\partial\theta^2}\right] \tag{24}$$

where dots represent derivatives with respect to $x$, i.e., $\dot{z} = \frac{dz}{dx}$, etc. The solutions to (17) then separate and have the form

$$F_m(\xi) = \Psi_m(x)e^{im\theta} \tag{25}$$

where $\Psi_m(x)$ satisfies the equation

$$\left\{-\frac{\partial^2}{\partial x^2} + \left(\frac{z\dot{z}}{1-z^2} + \frac{\ddot{z}}{\dot{z}}\right)\frac{\partial}{\partial x} + \sigma(\sigma+1)\frac{\dot{z}^2}{(1-z^2)^2}\right\}\Psi_m(x) = m^2\frac{\dot{z}^2}{1-z^2}\Psi_m(x) \tag{26}$$

which upon the substitution

$$z(x) = \sin x \tag{27}$$

transforms to the Schrödinger equation

$$H\Psi_m(x) = m^2\Psi_m(x) \tag{28}$$

with non-Hermitian Hamiltonian given by

$$H = -\frac{d^2}{dx^2} + \frac{\sigma(\sigma+1)}{\cos^2 x} \tag{29}$$

Moreover, it follows from (21) and (22) that

$$\begin{aligned} \Psi_m(x) &= \int \left|\tan x - \frac{\cos\varphi}{\cos x}\right|^{-1-\sigma} e^{im\varphi}d\varphi \\ &\equiv (\Omega\psi_m)(x) \end{aligned} \tag{30}$$

and

$$H\Omega = \Omega h \tag{31}$$

where

$$h = -J_3^2 \quad \text{and} \quad \psi_m = e^{im\varphi} \tag{32}$$

The verification of (31) is based on the relation

$$Hk(x,\varphi) = hk(x,\varphi) \tag{33}$$

where $k(x,\varphi)$ is the kernel of the intertwining operator $\Omega$, i.e.

$$k(x,\varphi) = \left|\tan x - \frac{\cos\varphi}{\cos x}\right|^{-1-\sigma} \tag{34}$$

Another example is provided by Hamiltonian of the form

$$H = -\frac{d^2}{dx^2} + \frac{\sigma(\sigma+1)}{\sin^2 x} \tag{35}$$

which is obtained by substituting

$$z = \cos x \tag{36}$$

In this case

$$\Psi_m(x) = \int \left|\cot x - \frac{\cos\varphi}{\sin x}\right|^{-1-\sigma} e^{im\varphi} \tag{37}$$

**References**


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