Complex trajectories of a simple pendulum

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Work based on:

- Carl M. Bender, Darryl D. Holm, Daniel W. Hook (2007) “Complex Trajectories of a Simple Pendulum”, Journal of Physics A40, F81-F89, [math-ph/0609068]
- Carl M. Bender, Darryl D. Holm, Daniel W. Hook (2007) “Complexified Dynamical Systems”, to appear: Journal of Physics A, [arXiv:0705.3893]
- Carl M. Bender, Jun-Hua Chen, Daniel W. Darg, Kimball A. Milton “Classical Trajectories for Complex Hamiltonians” (2006) Journal of Physics A39, 4219-4238 [math-ph/0602040]
- Carl M. Bender, Daniel W. Darg (2007) “Spontaneous Breaking of Classical PT Symmetry”, [hep-th/0703072]

The Simple Pendulum

Position:

$$\begin{aligned} X &= L\sin\theta \\ Y &= -L\cos\theta \end{aligned}$$

Velocity:

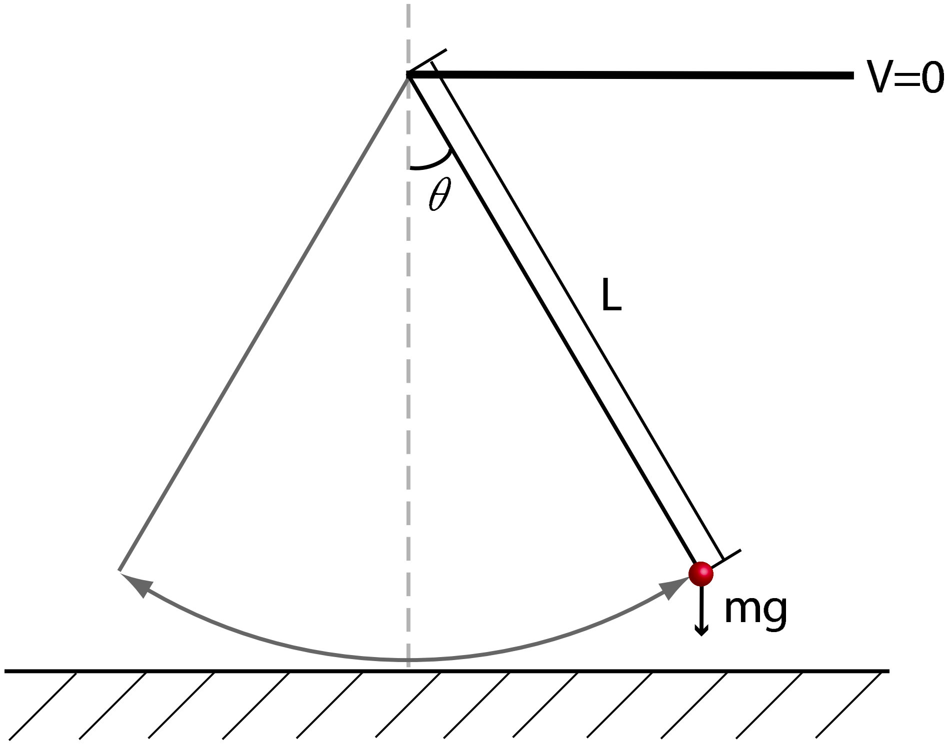
$$\begin{aligned} \dot{X} &= L\dot{\theta}\cos\theta \\ \dot{Y} &= L\dot{\theta}\sin\theta \end{aligned}$$

Energies:

$$\begin{aligned} V &= -mgL\cos\theta \\ T &= \tfrac{1}{2}m(\dot{X}^2 + \dot{Y}^2) = \tfrac{1}{2}mL^2\dot{\theta}^2 \end{aligned}$$

Hamiltonian:

$$H = \tfrac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta$$



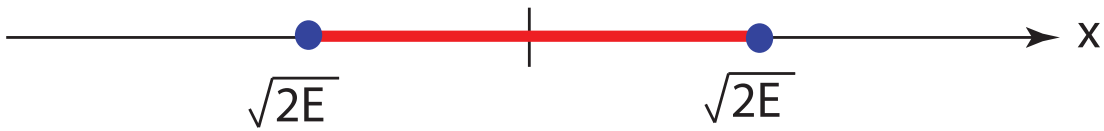
Turning points

We have turning points in the motion when

$$p = 0$$

Hence,

$$H = V(x) = E$$



For a simple harmonic oscillator with Hamiltonian $H = \frac{1}{2}p^2 + x^2$

$$x_0 = \sqrt{2E}$$

Complexifying the Simple pendulum

Gravity:

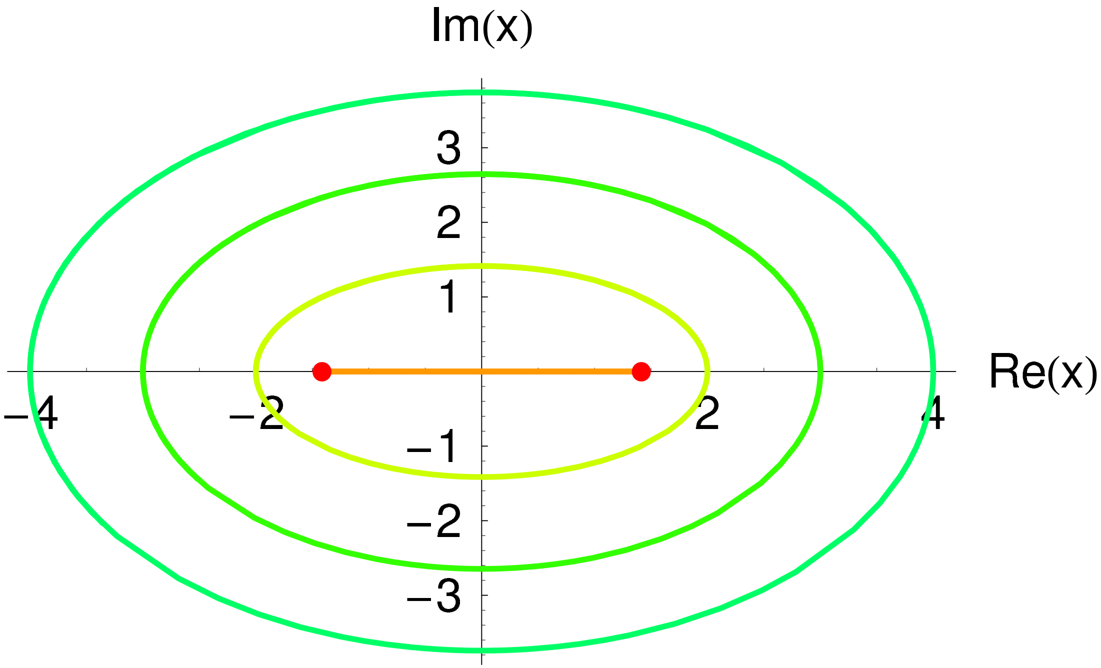
$$\mathbf{g = 1}$$

Turning points:

$$x_0 = \sqrt{2E}$$

There is a branch cut between turning points $(x = \sqrt{2E})$

Paths have the same period:
$T = \oint_C dx/\sqrt{2[E - V(x)]}$



A periodic potential

Case 1: $-1 \leq E \leq 1$ (Swinging pendulum)

Hamiltonian:

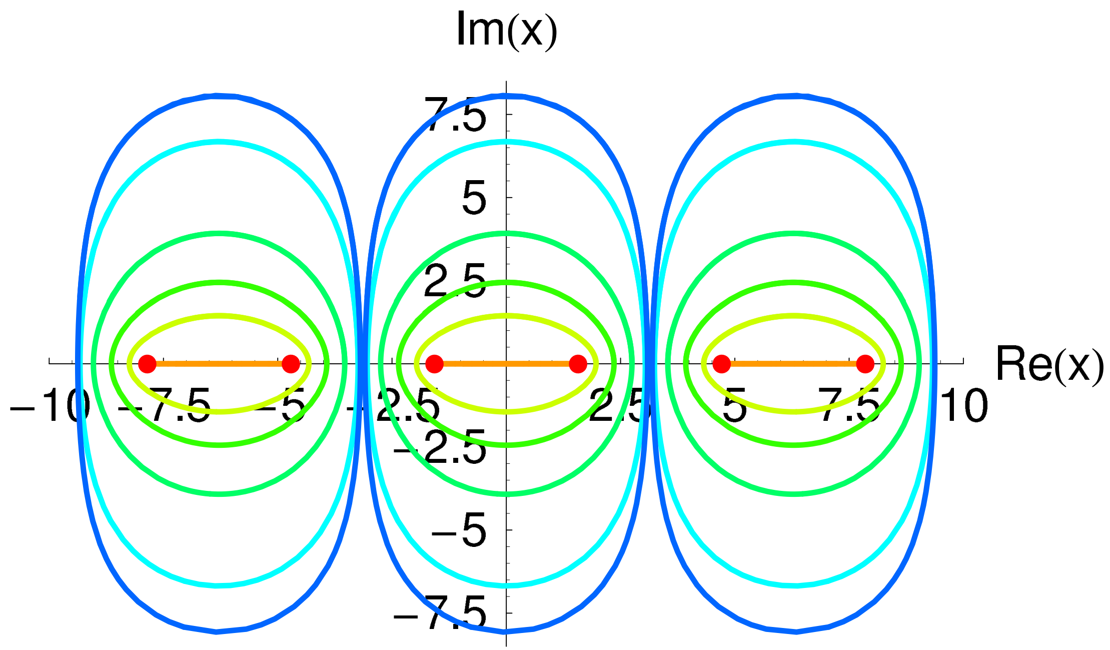
$$H = \frac{1}{2}p^2 - \cos x$$

Gravity:

$$\mathbf{g = 1}$$

Turning points:

$$x_0 = \pi/2 + n\pi$$



Example: Real Gravitational Field

Case 2: $|E| \geq 1$ (Rotating pendulum)

Hamiltonian:

$$H = \frac{1}{2}p^2 - \cos x$$

Gravity:

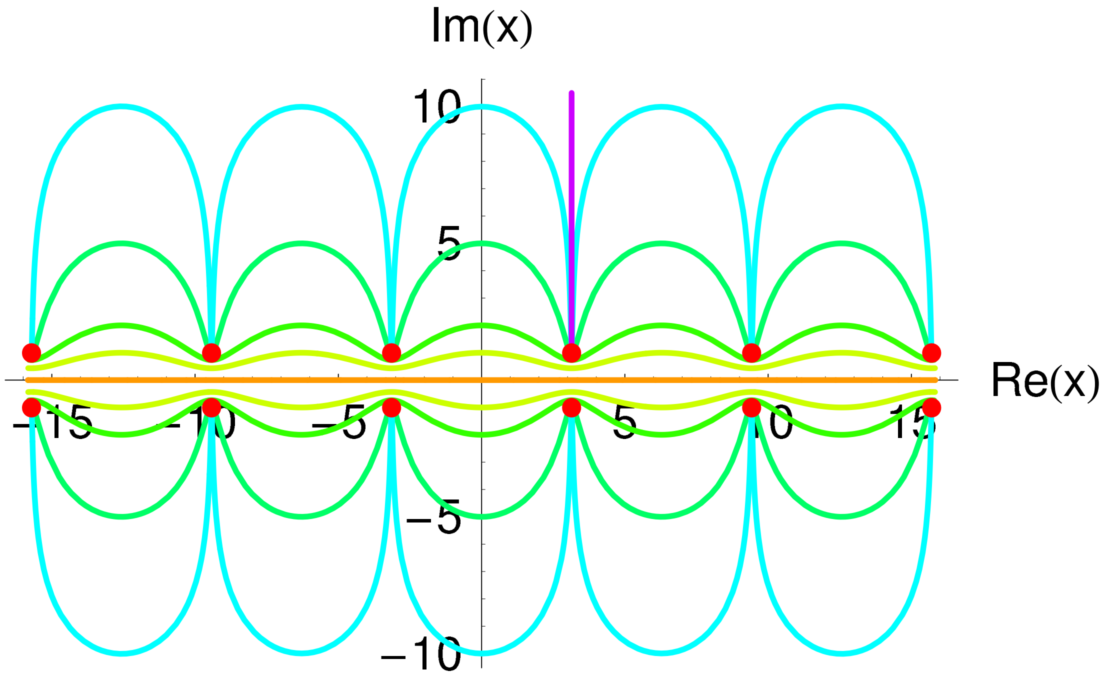
$$\mathbf{g = 1}$$

Energy:

$$\mathbf{E = \cosh 1}$$

Turning points:

$$x_0 = (2k+1)\pi \pm i$$

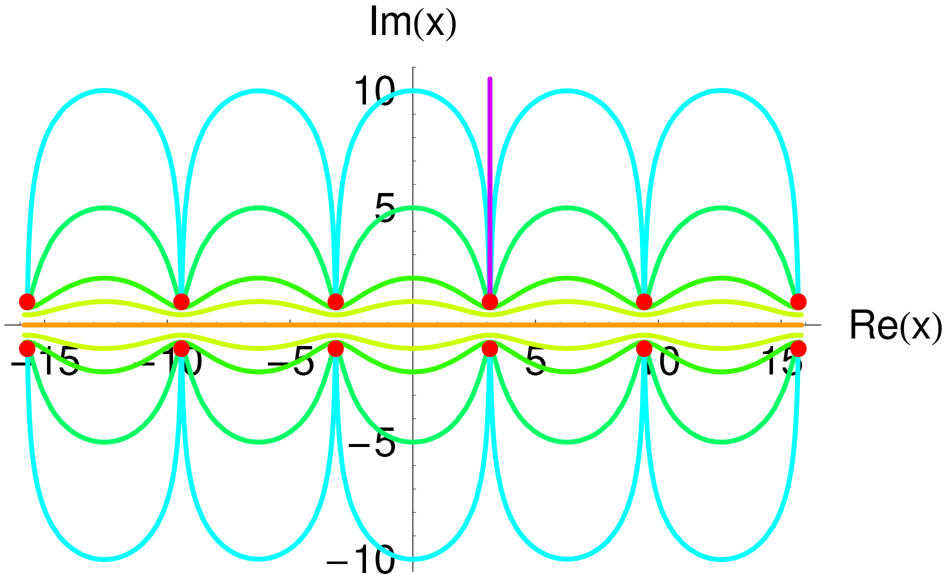


Example: Real Gravitational Field

Time to follow the purple path . . .

$$T = \frac{1}{\sqrt{2}} \int_{x=i+\pi}^{i\infty+\pi} \frac{dx}{\sqrt{E + \cos x}}$$

$$T = 1.97536 \cdots$$



Example: Real Gravitational Field

Case 3: $E \leq -1$ (Unphysical pendulum)

Hamiltonian:

$$H = \frac{1}{2}p^2 - \cos x$$

Gravity:

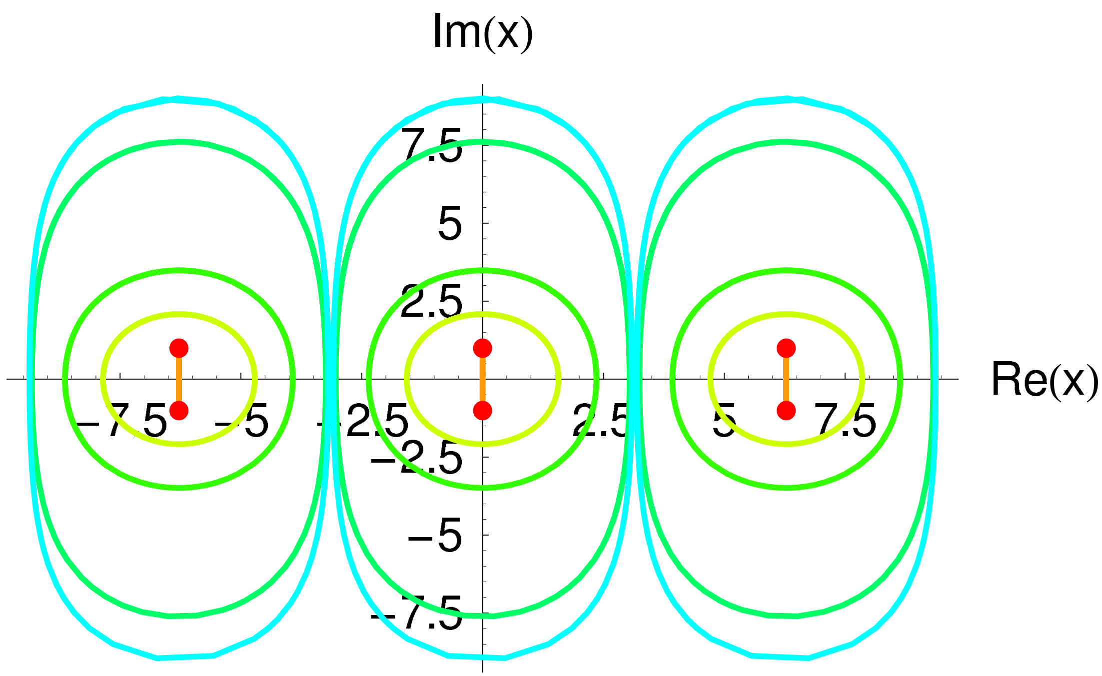
$$\mathbf{g = 1}$$

Energy:

$$\mathbf{E = -\cosh 1}$$

Turning points:

$$x_0 = 2\pi k + \pm i$$



Example: Imaginary Gravitational Field

Case 1: $E = \sinh 1$

Hamiltonian:

$$H = \frac{1}{2}p^2 - i\cos x$$

Gravity:

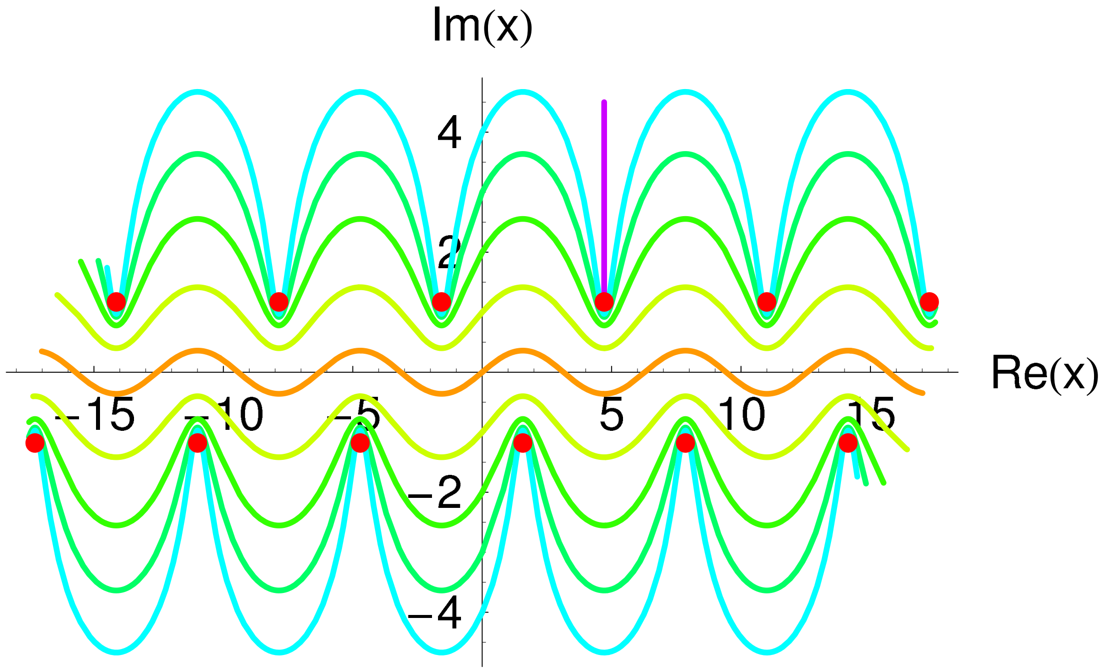
$$\mathbf{g} = \mathbf{i}$$

Energy:

$$\mathbf{E} = \sinh \mathbf{1}$$

Turning points:

$$x_0 = (n + \tfrac{1}{2})\pi + (-1)^n i$$



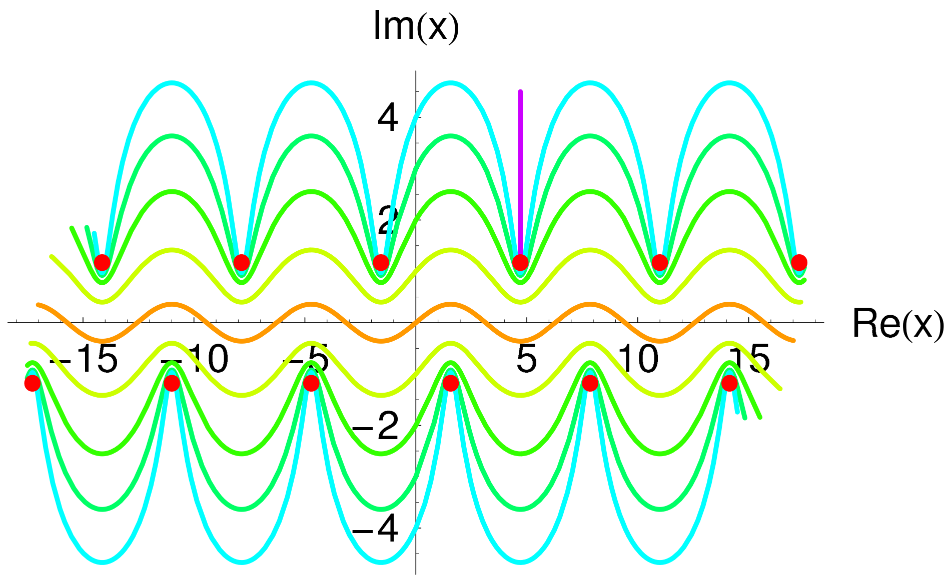
Example: Imaginary Gravitational Field

Time to follow the purple path . . .

$$T = \frac{1}{\sqrt{2}} \int_{x=i+3\pi/2}^{i\infty+3\pi/2} \frac{dx}{\sqrt{E + i\cos x}}$$

$$T = \frac{1}{\sqrt{2}} \int_{s=1}^{\infty} \frac{ds}{\sqrt{\sinh s - \sinh 1}}$$

$$= \frac{2}{\sqrt{e}} K(-1/e^2) = 1.84549\cdots$$



Example: Imaginary Gravitational Field

Case 2: $E = -\sinh 1$

Hamiltonian:

$$H = \frac{1}{2}p^2 - i\cos x$$

Gravity:

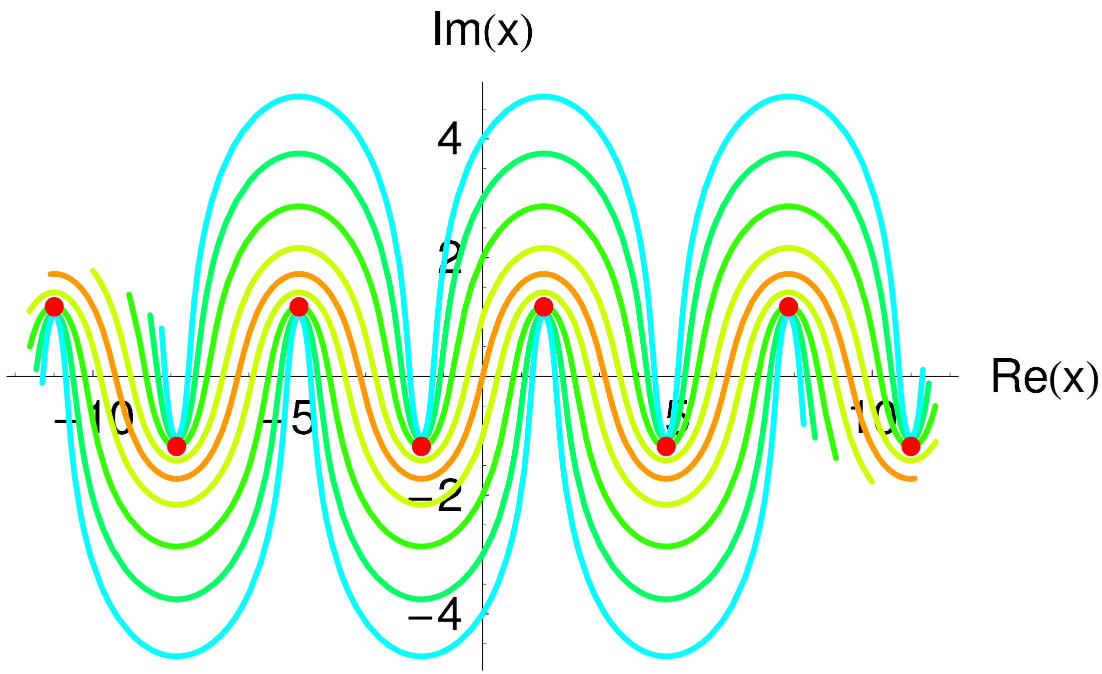
$$\mathbf{g} = \mathbf{i}$$

Energy:

$$\mathbf{E} = -\sinh \mathbf{1}$$

Turning points:

$$x_0 = (n+\frac{1}{2})\pi + (-1)^{n+1} i$$

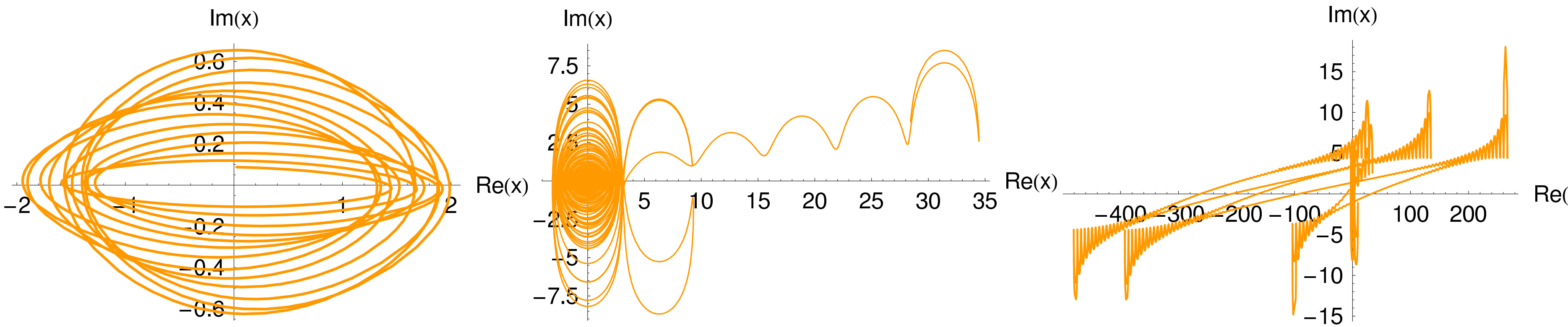


What about chaotic systems?

Introduce a driving term of the form $\epsilon \sin(\omega t)$:

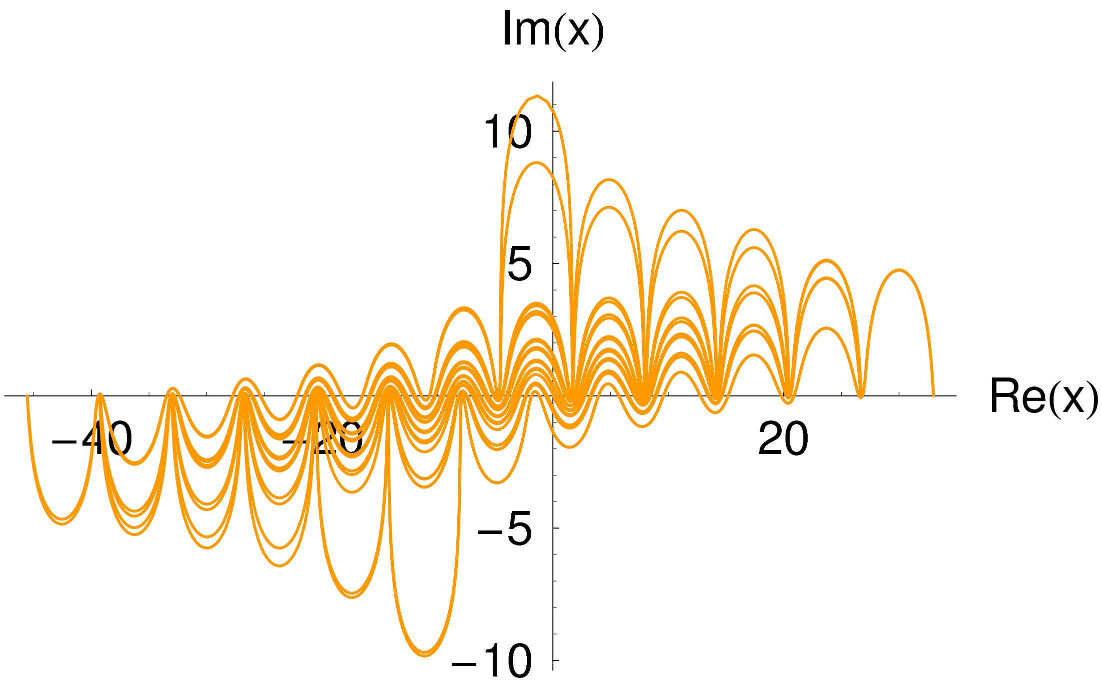
$$H = \frac{1}{2}p^2 - g\cos x - \epsilon \sin(\omega t)$$

$\mathbf{g = i}$, $\epsilon > \mathbf{0}$



Imaginary energy?

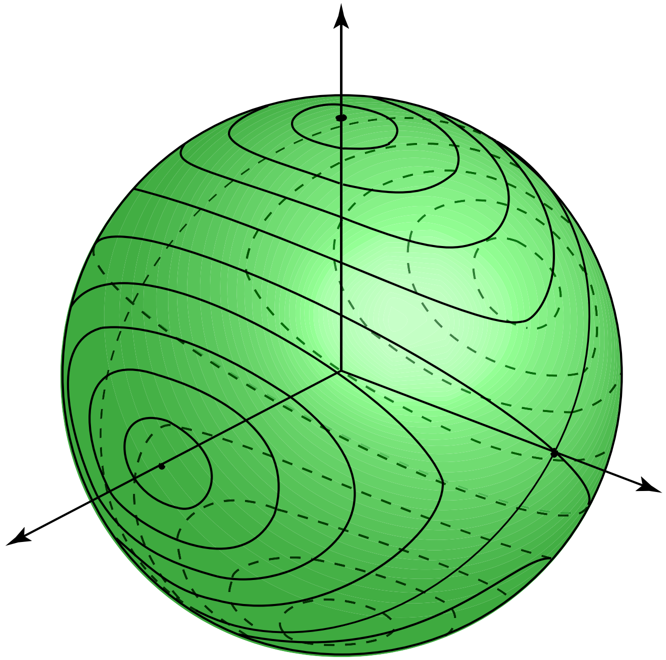
$g = i$ and $E = i$



Euler Equations

The Euler equations govern-
ing the rotation of a rigid body about a fixed axis also are $\mathcal{PT}$.

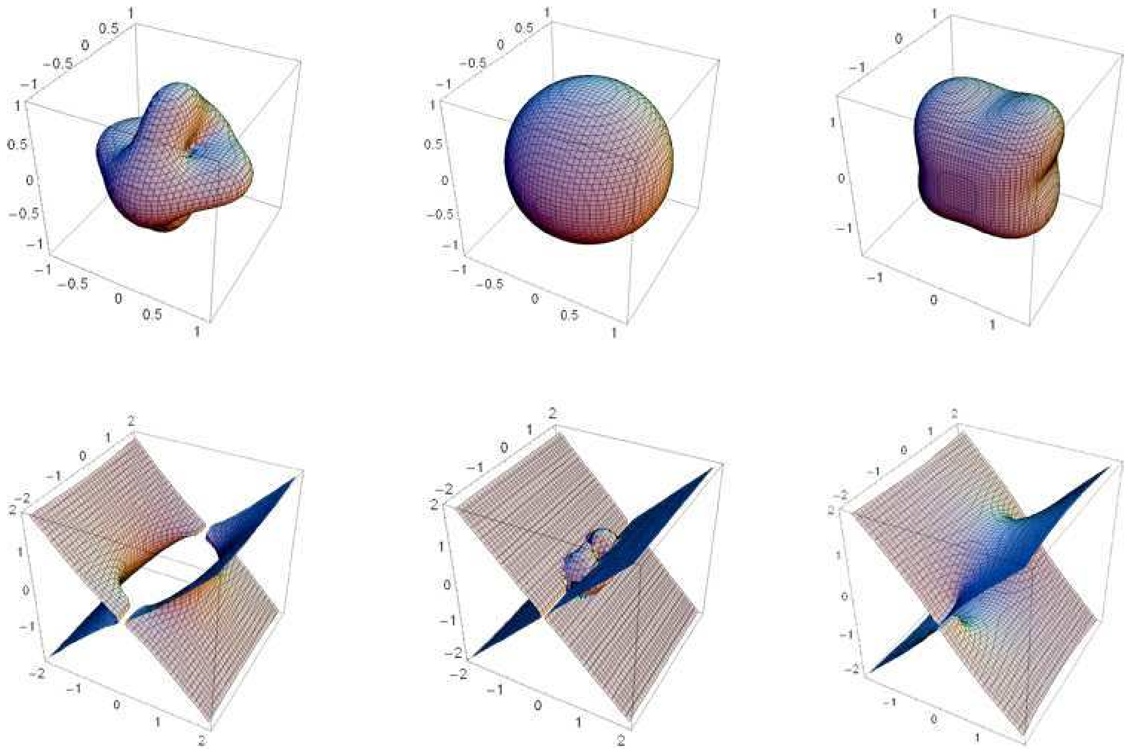
$$\begin{aligned} \dot{L}_1 &= L_2 L_3 \\ \dot{L}_2 &= -2 L_1 L_3 \\ \dot{L}_3 &= L_1 L_2 \end{aligned}$$



Euler Equations

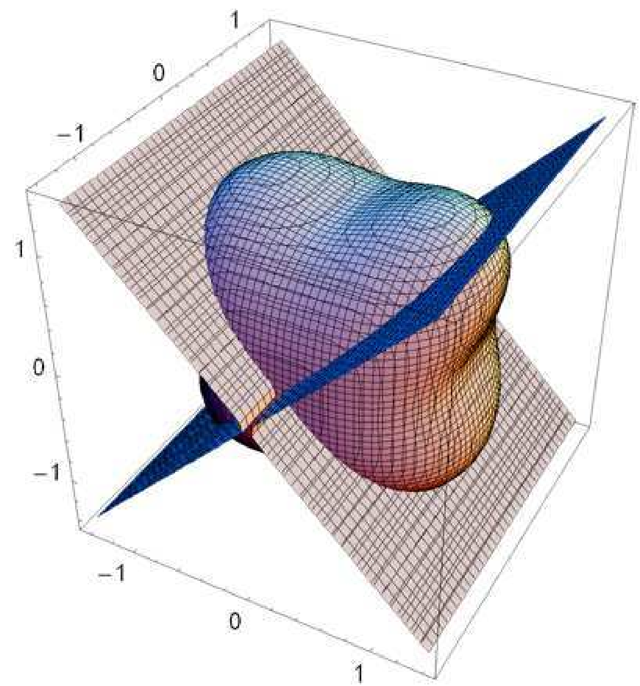
We extend the Euler Equations into the complex plane. There are two constants of motion (which gives us 4 constraints):

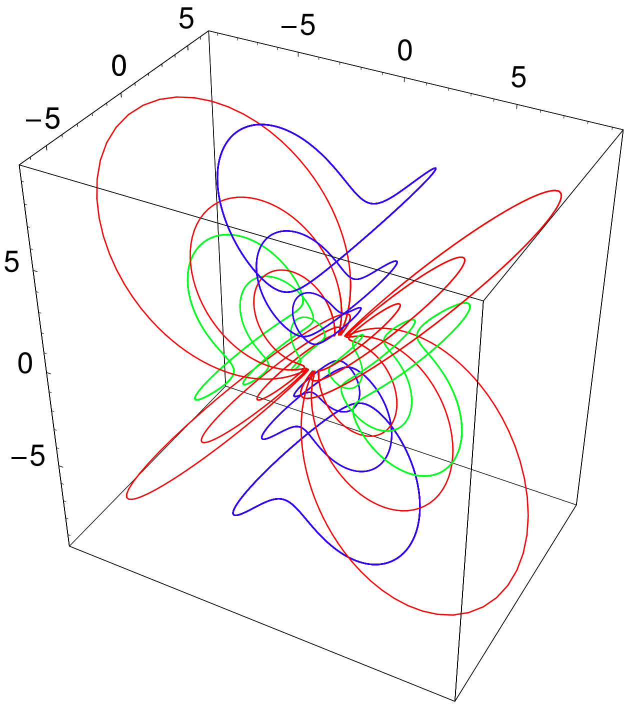
$$C = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) \qquad \text{and} \qquad H = \frac{1}{2}x_3^2 - \frac{1}{2}x_1^2.$$



Euler Equations

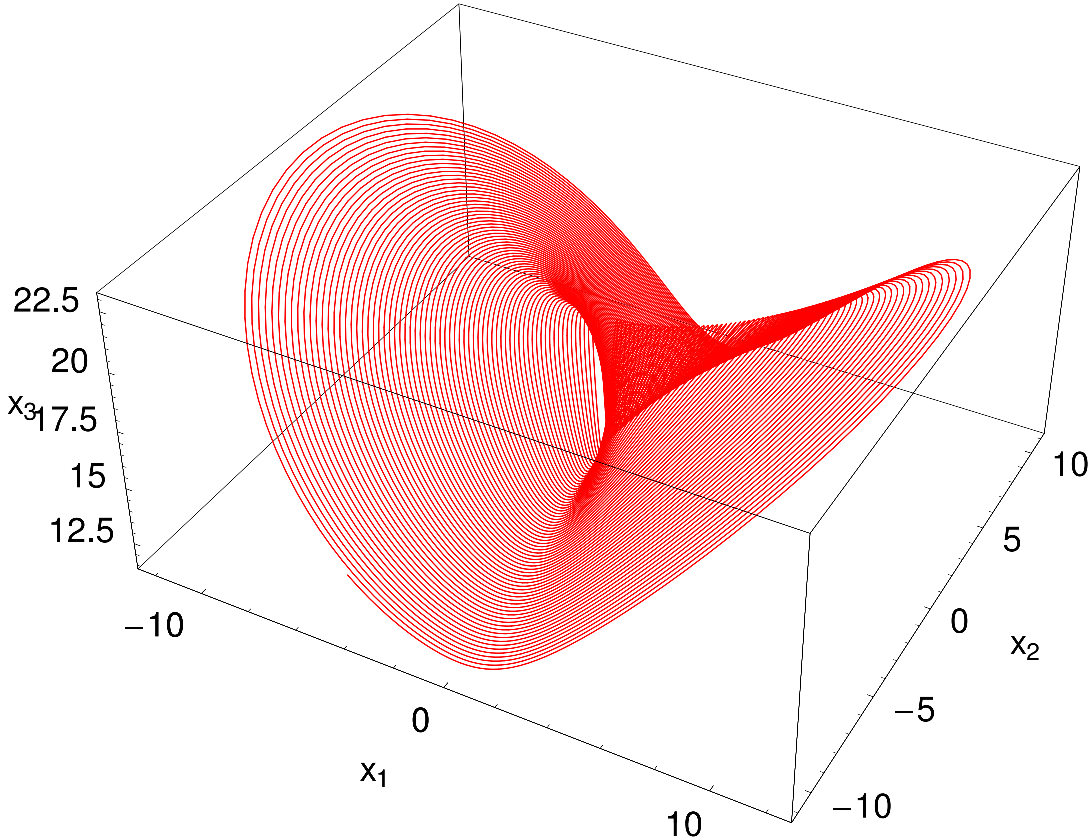
Intersections of the level surfaces give us the orbits:





Euler Equations

If we allow energy to be imaginary:



Lotka-Volterra equations

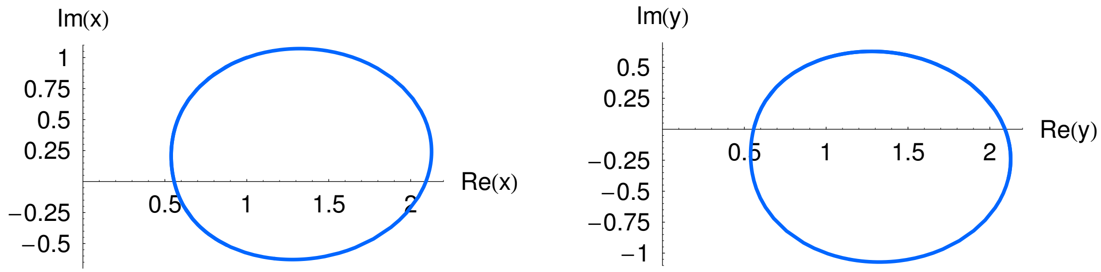
Another PT-Symmetric system - this time under:

$$\dot{x} = x - xy, \quad \dot{y} = -y + xy \tag{1}$$

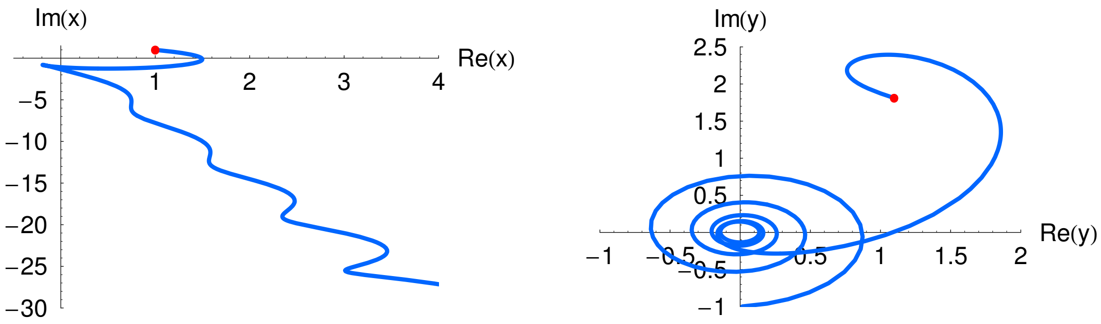
constant of motion

$$x + y - \log(xy) = C. \tag{2}$$

Fox/ rabbit interchange, and time reversal.



Lotka-Volterra equations



Conclusions

- Classical Mechanics seems to naturally extend into the complex plane to give some familiar and not so familiar results
- Complex extensions of classical mechanics help us understand well known systems

Where next?

- spherical pendulum, the spinning top
- other discrete dynamical systems - SIR models etc.
- classification of chaotic systems using $\mathcal{PT}$ Symmetry (see for example, the Kicked Rotor).