Interfacing the Hermitian and Pseudo-Hermitian Worlds

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Outline

1. Introduction

Development of subject:

- Initial discovery of real eigenvalues. Exploration of soluble models

- Need for +ve. definite metric. CPT , η

- Construction of equivalent Hermitian Hamiltonian h

- Ghost busting: Lee model, Pais-Uhlenbeck model

All above concerns non-Hermitian systems in isolation

But most of physics is Hermitian.

$\therefore$ have to consider interface between two.

First attempts:

- Quantum Brachistochrone (CMB et al.)

- Coupling Hermitian and non-Hermitian Hamiltonians (CMB
HFJ)

- Scattering off localized complex potentials (HFJ)

2. Quantum Brachistochrone

For Hermitian Hamiltonians $\exists$ lower bound on "passage time"

= time for (unitary) evolution by e^{-iHt} between orthogonal states

For 2×2 matrices, and fixed dispersion $E_+ - E_- = \omega$, bound

$\left[\text{e.g. from} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{to} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$ is

$$\boxed{t \geq \pi/\omega}$$

Can we do better using a non-Hermitian H?

Yes!

Take

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}$$

Real eigenvalues when $r \sin\theta < s$. So write $r \sin\theta = s \sin\alpha$
Then $\omega = 2s \cos\alpha$, and passage time is

$$\boxed{t = (\pi + 2\alpha)/\omega}$$

Can be made arbitrarily small as $\alpha \to -\pi/2$!

How do we avoid theorem?

- In framework of conventional QM, states are orthogonal, but H is not Hermitian, and time evolution is not unitary.

- Alternatively, to describe transition can introduce pseudo-Hermitian metric η . Then evolution is unitary w.r. to η , but states are not orthogonal.

Note interface between Hermitian world (initial and final states) and intervening non-Hermitian Hamiltonian.

3. Coupling Hermitian and non-Hermitian Hamiltonians

Here we consider

$$H = H_1 + H_2 + \text{coupling},$$

where H_1 is Hermitian, and H_2 is quasi-Hermitian

3.1 **Matrix model:**

$$H = \left(\begin{array}{cc|cc} 1 & 1 & \varepsilon & 0 \\ 1 & 1 & 0 & \varepsilon \\ \hline \varepsilon & 0 & re^{i\theta} & s \\ 0 & \varepsilon & s & re^{-i\theta} \end{array}\right)$$

Unperturbed eigenvalues are 0 , 2 , $r\cos\theta \pm s\cos\alpha$
($r\sin\theta = s\sin\alpha$)

Numerically, eigenvalues are real up to $\varepsilon \approx 0.704$. Then complex.

3.2 **SHO+shifted SHO:**

$$H = \underbrace{p^2 + x^2}_{H_1} + \underbrace{q^2 + y^2 + 2iy}_{H_2} + \underbrace{2\varepsilon xy}_{\text{coupling}}$$

Construct $\mathcal{Q}$ to satisfy

$$H^\dagger = e^{-\mathcal{Q}} H e^{\mathcal{Q}}$$

Solution is $$\mathcal{Q} = 2\left(\frac{\varepsilon p - q}{1 - \varepsilon^2}\right)$$

Note possible problems when $|\varepsilon| \to 1$

Now construct equivalent Hermitian Hamiltonian:

$$\begin{aligned} h &= e^{-\frac{1}{2}\mathcal{Q}} H e^{\frac{1}{2}\mathcal{Q}} \\ &= p^2 + x^2 + q^2 + y^2 + 2\varepsilon xy + \frac{1}{1-\varepsilon^2} \end{aligned}$$

Two coupled SHOs. Diagonalize by

$$x = \frac{1}{\sqrt{2}}(X+Y) \qquad p = \frac{1}{\sqrt{2}}(P+Q)$$
$$y = \frac{1}{\sqrt{2}}(X-Y) \qquad q = \frac{1}{\sqrt{2}}(P-Q)$$

Net result is

$$h = P^2 + (1+\varepsilon)X^2 + Q^2 + (1-\varepsilon)Y^2 + \frac{1}{1-\varepsilon^2}$$

with eigenvalues

$$E_{m,n} = (2m+1)\sqrt{1+\varepsilon} + (2n+1)\sqrt{1-\varepsilon} + \frac{1}{1-\varepsilon^2}$$

So eigenvalues complex for $\boxed{|\varepsilon| > 1}$

3.3 **SHO+Swanson Hamiltonian:**

$$H = (p^2 + x^2) + (q^2 + y^2 + ic\{q, y\}_+) + 2\varepsilon xy$$

Can take $\mathcal{Q} = -cy^2$, which shifts $q \to q - icy$

Then

$$h = p^2 + x^2 + q^2 + (1 - c^2)y^2 + 2\varepsilon xy$$

Can be diagonalized, to give

$$h = P^2 + \Omega_1^2 X^2 + Q^2 + \Omega_2^2 Y^2$$

where

$$\Omega_{1,2} = 1 - \frac{1}{2}c^2 \pm \left(\varepsilon^2 + \frac{1}{4}c^4\right)^{\frac{1}{2}}$$

Eigenvalues

$$E_{m,n} = (2m+1)\Omega_1 + (2n+1)\Omega_2$$

again become complex when $\boxed{\varepsilon^2 > 1 - c^2}$

3.4 **Generic Real $V(x)$+shifted SHO:**

$$H = (p^2 + V(x)) + (q^2 + y^2 + 2iy) + 2\varepsilon xy$$

Can show in perturbation theory that E real up to $O(\varepsilon^2)$.
Attempting proof that E becomes complex for some ε

4. Conceptual Problems in Scattering

4.1 **Scattering off Simple non-Hermitian Potentials**

(i) Two delta functions

Consider PT-symmetric potential:

$$V(x) = i\lambda(\delta(x-a) - \delta(x+a)) \tag{4.1}$$

with WF

$$\psi = \begin{cases} e^{ikx} + Ce^{-ikx} & x < -a \\ Ae^{ikx} + Ce^{-ikx} & -a < x < a \\ De^{ikx} & a < x \end{cases} \tag{4.2}$$

Applying cont^y cond^{ns} $[\psi] = 0$, $[\psi'] = \pm i\lambda\psi$ at boundaries, get

$$D = \frac{1}{1 + i\alpha^2 e^{2ika} \sin 2ka}$$

$$C = 2iD\alpha(1-\alpha)\sin 2ka,$$

$(\alpha \equiv \lambda/(2k))$

In particular

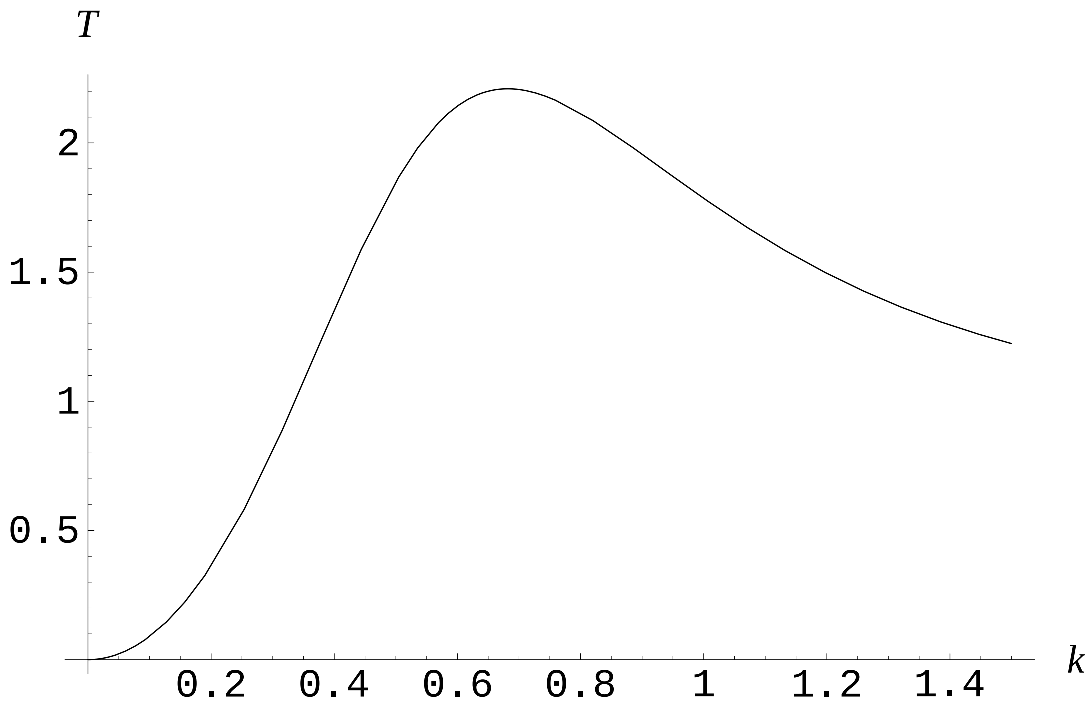
$$|D|^2 = \frac{1}{1 - 4\alpha^2(1-\alpha^2)\sin^2 2ka}$$

$$> 1 \quad \text{for} \quad \alpha < 1$$

But in conventional QM, $|D|^2$ represents the transmission probability . So probability not conserved even though V is PT-symmetric.

(ii) Same is true for complex square well (PT-symmetric):

$$V(x) = \begin{cases} 0 & |x| > a \\ -i\lambda & -a < x < 0 \\ i\lambda & 0 < x < a \end{cases}$$



(iii) and for single complex δ function: $V = z\delta(x)$,
with $z = 2\lambda(1 + i\varepsilon)$
WF is just

$$\psi = \begin{cases} e^{ikx} + Ce^{-ikx} & x < 0 \\ \\ De^{ikx} & 0 < x, \end{cases}$$

with

$$D = \left(1 + \frac{iz}{2k}\right)^{-1}, \quad C = -\frac{iz}{2k} D \ .$$

Now

$$|C|^2 + |D|^2 \ = \ \left(1 - \frac{2\varepsilon q}{1 + \varepsilon^2 + q^2}\right)^{-1}$$

$$\gtrless \ 1 \quad \text{for} \ \ \varepsilon \gtrless 0$$

$(k = \lambda q)$

So in each case unitarity (as conventionally calculated) is violated.

Two approaches:

1. Treat V as an <u>effective</u> pot$^{\ell}$, and don't worry (Cannata et al.)

2. Treat V as <u>fundamental</u>, and use appropriate η metric

Will go through this exercise, but note from beginning that it must involve a fundamental redefinition of QM even at infinity.

4.2 **Quasi-Hermitian Approach ($V = z\delta(x)$)** (AM)

Recall that metric $\eta \equiv e^{-Q}$ is defined by

$$H^\dagger = \eta H \eta^{-1}$$

Then calculate matrix elements by including η
e.g.

$$\langle A \rangle_\psi = \langle \psi | \eta A | \psi \rangle$$

A is an observable, with real eigenvalues, if it is quasi-Hermitian:
$A^\dagger = \eta A \eta^{-1}$

Mostafazadeh has calculated perturb$^{\text{n}}$ series for η :

$$\eta = \sum_{r=0}^{\infty} \varepsilon^r \eta^{(r)}$$

up to $O(\varepsilon^3)$.

Matrix elements of $\eta^{(0)}$ and $\eta^{(1)}$ are

$$\eta^{(0)}(x,y) = \delta(x-y)$$

$$\eta^{(1)}(x,y) = \frac{1}{2} i\lambda[\theta(xy)e^{-\lambda|x-y|} + \theta(-xy)e^{-\lambda|x+y|}] \, \mathrm{sgn}(y^2 - x^2)$$

If we use η metric, x is no longer an observable. Instead position observable is X, defined by

$$X = \rho^{-1} x \rho,$$

where $\rho = \eta^{\frac{1}{2}} = e^{-\frac{1}{2}Q}$

Big problem is that η and ρ are non-local .

According to AM, relevant WF is not $\psi(x) \equiv \langle x|\psi\rangle$, but

$$\Psi(x) \equiv \langle x|\Psi\rangle = \langle x|\rho|\psi\rangle$$

Then

$$\begin{aligned}\langle \widehat{X}\rangle_\psi &= \langle\psi|\eta\widehat{X}|\psi\rangle \\ &= \langle\Psi|\rho^{-1}\eta\ (\rho^{-1}\widehat{x}\rho)\rho^{-1}|\Psi\rangle \\ &= \langle\Psi|\widehat{x}|\Psi\rangle = \int x|\Psi(x)|^2 dx\end{aligned}$$

Take new probability density as $\varrho \equiv |\Psi(x)|^2$.
Then total probability is conserved in time:

$$d(\int \varrho\ dx)/dt = 0$$

[But $\nexists$ local conservation equation of form $\partial\varrho/\partial t + dj/dx = 0$]

So, have to calculate

$$\Psi(x) = \int dy \rho(x,y) \psi(y),$$

where $\rho = \eta^{(0)} + \frac{1}{2}\varepsilon\eta^{(1)} + O(\varepsilon^2)$

Recall that

$$\psi = \begin{cases} e^{ikx} + Ce^{-ikx} & x < 0 \\ \\ De^{ikx} & 0 < x, \end{cases}$$

with

$$D = \left(1 + \frac{iz}{2k}\right)^{-1}, \quad C = -\frac{iz}{2k}D \ .$$

Result of this calculation for $x > 0$, and to $O(\varepsilon)$, is

$$\Psi_>(x) = De^{ikx} + \frac{\varepsilon\lambda k}{2(\lambda^2+k^2)}\left(e^{-ikx} - (C+D)e^{ikx}\right) + O(e^{-\lambda x})$$

N.B. $\Psi_>(x)$ no longer represents a pure outgoing wave $\propto e^{ikx}$
Contains term $\propto e^{-ikx}$ as well. $\therefore$ physical picture of the scattering is completely changed.

But can neglect it to calculate probabilities to $O(\varepsilon)$
Then get

$$\Psi_>(x) = e^{ikx}\frac{q}{q+i}\left(1 + \frac{\varepsilon}{2(q+i)}\right)$$

$(k = q\lambda)$

Similarly, for $x < 0$, get

$$\Psi_<(x) = e^{ikx}\left(1 + \frac{\varepsilon q}{2(q^2+1)}\right) - \frac{ie^{-ikx}}{q+i}\left(1 + \frac{\varepsilon}{2(q+i)} - i\varepsilon\right)$$

so that

$$|\Psi_<(x)|^2 \ = \ \underbrace{\left(1 + \frac{\varepsilon q}{q^2+1}\right)}_{\text{incoming flux}} + \underbrace{\frac{1}{q^2+1}\left(1 + \frac{\varepsilon q}{q^2+1}\right)}_{\text{outgoing flux}} + \text{interference term}$$

So just multiply Hermitian fluxes for real δ function $2\lambda\delta(x)$ by same common factor.

Hence newly-defined probability is indeed conserved to this order

5. Summary

1. Quantum brachistochrone works because of mixture of Hermitian and non-Hermitian Hamiltonians.

2. Can couple Hermitian and non-Hermitian systems weakly while retaining real energies.

3. Scattering presents quasi-Hermitian QM with a quandary. If Hamiltonian is to be treated as fundamental it necessitates a change in the framework of QM even at infinity.

   N.B. Problem is generic, and not confined to δ-functions and square wells.