Dynamics of open quantum systems
described by
a non-Hermitian Hamilton operator

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Closed quantum system with discrete eigenstates (standard quantum mechanics)

$$(H^B - E^B_\lambda)\Phi^B_\lambda = 0$$

H^B hermitian ; E^B_λ real

Φ^B_λ real (up to a common phase)

$$\longrightarrow \langle \Phi^B_\lambda | \Phi^B_{\lambda'} \rangle = \delta_{\lambda\lambda'}$$

Open quantum system with resonance states

$$(H - z_\lambda)\Phi_\lambda = 0$$

H non-hermitian; $z_\lambda = E_\lambda - \frac{i}{2}\Gamma_\lambda$ complex

Φ_λ complex and biorthogonal

$$\longrightarrow \langle \Phi^*_\lambda | \Phi_{\lambda'} \rangle = \delta_{\lambda\lambda'}$$

The system with N states ("Q subsystem") is embedded in the continuum of scattering wave functions ξ^E_C ("P subsystem")

$$\longrightarrow \quad P + Q = 1$$

The eigenfunctions of the non-hermitian (symmetric) H are bi-orthogonal

$$\langle \Phi_\lambda^* | \Phi_{\lambda'} \rangle = \delta_{\lambda\lambda'}$$

A consequence of the bi-orthogonality relations

$$\begin{aligned} \langle \Phi_\lambda | \Phi_\lambda \rangle &= \mathrm{Re}\left(\langle \Phi_\lambda | \Phi_\lambda \rangle \right) \\ \langle \Phi_\lambda | \Phi_{\lambda' \neq \lambda} \rangle &= \mathrm{i} \cdot \mathrm{Im}\left(\langle \Phi_\lambda | \Phi_{\lambda' \neq \lambda} \rangle \right) = -\langle \Phi_{\lambda' \neq \lambda} | \Phi_\lambda \rangle \end{aligned}$$

with

$$\begin{aligned} A_\lambda &\equiv \langle \Phi_\lambda | \Phi_\lambda \rangle \geq 1 \\ |B_\lambda^{\lambda' \neq \lambda}| &\equiv |\langle \Phi_\lambda | \Phi_{\lambda' \neq \lambda} \rangle| \geq 0 \end{aligned}$$

Another consequence of the normalization of the bi-orthogonal wave functions Φ_λ

$$\begin{aligned}\langle \Phi_\lambda^* | \Phi_\lambda \rangle &= (\Phi_\lambda)^2 = (\mathrm{Re}\Phi_\lambda + i\ \mathrm{Im}\Phi_\lambda)^2 \\ &= (\mathrm{Re}\Phi_\lambda)^2 - (\mathrm{Im}\Phi_\lambda)^2 + 2\ i\ \mathrm{Re}\Phi_\lambda \cdot \mathrm{Im}\Phi_\lambda\end{aligned}$$

$$\langle \Phi_\lambda^* | \Phi_\lambda \rangle = 1 \ \longrightarrow\ \mathrm{Re}\Phi_\lambda \cdot \mathrm{Im}\Phi_\lambda = 0$$

The condition $\mathrm{Re}\Phi_\lambda \cdot \mathrm{Im}\Phi_\lambda = 0$ corresponds to a

rotation

(phase change) of the Φ_λ

Phase rigidity of the eigenfunctions of H

$$r_\lambda = \frac{\langle \Phi_\lambda^* | \Phi_\lambda \rangle}{\langle \Phi_\lambda | \Phi_\lambda \rangle} = \frac{1}{(\mathrm{Re}\Phi_\lambda)^2 + (\mathrm{Im}\Phi_\lambda)^2} = \frac{1}{A_\lambda}$$

Small coupling via the environment

$$\longrightarrow r_\lambda \approx 1$$

Approaching a crossing point

$$\longrightarrow r_\lambda \to 0$$

Therefore

$$0 \le r_\lambda \le 1$$

Alignment

of the wave functions Φ_λ (resonance states) with the wave functions ξ_C^E (scattering states) by means of varying r_λ

$$r_\lambda = 1 \quad \longrightarrow \quad \mathrm{Im}\Phi_\lambda = 0$$

$$\text{isolated resonances}$$

$$r_\lambda = 0 \quad \longrightarrow \quad |\mathrm{Im}\Phi_\lambda| = |\mathrm{Re}\Phi_\lambda|$$

$$\text{crossing of resonances}: \Phi_1 \to \pm\, i\, \Phi_2$$

The transition is experimentally proven:
Dembowski et al., PRL **90**, 034101 (2003)

Enhancement of observables

$$\langle \Phi_\lambda^* | X | \Phi_\lambda \rangle = x_\lambda \;\longrightarrow\; \langle \Phi_\lambda | X | \Phi_\lambda \rangle = \frac{x_\lambda}{r_\lambda} \geq x_\lambda$$

Example

$$\Gamma_\lambda \quad \leq \quad \sum_C (\gamma_\lambda^C)^2$$

threshold effect (one-channel case)

$$\Gamma_\lambda \quad \leq \quad (\gamma_\lambda^C)^2$$

Unified description of structure and reaction

– Feshbach projection operator technique –

Schrödinger equation in the whole function space

$$(H^{\text{full}} - E)\Psi_E^c = 0$$

$$
\begin{aligned}
H^{\text{full}} &= H_{QQ} + H_{QP} + H_{PQ} + H_{PP} \\
(H_B &- E_\lambda^B)\,\Phi_\lambda^B = 0 \\
&\longrightarrow \; Q = \sum_\lambda |\Phi_\lambda^B\rangle\langle\Phi_\lambda^B| \\
(H_c &- E)\,\xi_c^{E(+)} = 0 \\
&\longrightarrow \; P = \sum_c \int_{\epsilon_c}^{\epsilon_c'} dE\; |\xi_c^{E(+)}\rangle\langle\xi_c^{E(+)}| \\
\gamma_{\lambda c}^0 &= \sqrt{2\pi}\,\langle\Phi_\lambda^B|H_{QP}|\xi_c^{E(+)}\rangle \\
Q &+ P = 1 \\
&(\Psi_c^E \text{ contains everything})
\end{aligned}
$$

H^{full} is hermitian

Solution of the Schrödinger equation in the whole function space (according to Feshbach)

$$\Psi_c^E = \xi_c^E + \sum_{\lambda,\lambda'=1}^N (\Phi_\lambda^B + \omega_\lambda^0) \langle \Phi_\lambda^B | \frac{1}{E-H_{\rm eff}} | \Phi_{\lambda'}^B \rangle \langle \Phi_{\lambda'}^B | H_{QP} | \xi_c^E \rangle$$

$$H_{\rm eff} = H_{QQ} + H_{QP} G_P^{(+)} H_{PQ}$$

$$\omega_\lambda^{0(+)} = G_P^{(+)} H_{PQ} \cdot \Phi_\lambda^B$$

$$G_P^{(+)} = P(E - H_{PP})^{-1} P$$

$$\Psi_c^E = \xi_c^E + \sum_{\lambda=1}^N \Omega_\lambda \cdot \frac{\langle \Phi_\lambda^* | H_{QP} | \xi_c^E \rangle}{E - z_\lambda}$$

$$\Omega_\lambda = (1 + G_P^{(+)} H_{PQ}) \Phi_\lambda = (1 + \omega_\lambda) \Phi_\lambda$$

$H_{\rm eff}$ is non-hermitian

Solution is exact within $P + Q = 1$

Scattering wave function inside the system

$$\Psi_c^E \to \Psi_{c\ \mathrm{in}}^E = \sum_{\lambda=1}^{N} c_{\lambda E}\ \Phi_\lambda$$

The Ψ_c^E are represented in a
set of biorthogonal wave functions $\{\Phi_\lambda\}$

$$\begin{aligned} |\Psi_{c\ \mathrm{in}}^{E,R}\rangle &= \sum_\lambda c_{\lambda E} |\Phi_\lambda^R\rangle = \sum_\lambda c_{\lambda E} |\Phi_\lambda\rangle \\ \langle\Psi_{c\ \mathrm{in}}^{E,L}| &= \sum_\lambda c_{\lambda E}^* \langle\Phi_\lambda^L| = \sum_\lambda c_{\lambda E}^* \langle\Phi_\lambda^*| \end{aligned}$$

$$c_{\lambda E} = \frac{\langle\Phi_\lambda^* | H_{QP} | \xi_c^E\rangle}{E - z_\lambda}$$

Phase rigidity of the Ψ_c^E at energy E

$$\rho = e^{2i\Theta}\frac{\langle\Psi_{\rm in}^{E,L*}|\Psi_{\rm in}^{E,R}\rangle}{\langle\Psi_{\rm in}^{E,L}|\Psi_{\rm in}^{E,R}\rangle} = e^{2i\Theta}\langle\Psi_{\rm in}^{E,L*}|\Psi_{\rm in}^{E,R}\rangle$$

$$= e^{2i\Theta}\sum_{\lambda\lambda'} c_{\lambda E}\, c_{\lambda' E}\langle\Phi_\lambda|\Phi_{\lambda'}\rangle$$

$$= e^{2i\Theta}\sum_{\lambda}\frac{(c_{\lambda E})^2}{r_\lambda}$$

$$0 \le \rho \le 1$$

Alignment of the scattering wave function Ψ_c^E

with the channel wave function $\xi_c^E \;\; (c = 1, ..., K)$

$\longrightarrow$ resonance trapping:

all but K resonance states are

hierarchically decoupled

(trapped) from the continuum

The resonance trapping phenomenon is proven experimentally: Persson et al., PRL **85**, 2478 (2000)

Alignment of resonance states (and resonance trapping) is a

collective phenomenon

to which all resonance states in a large energy region contribute

see, e.g., Jung et al., PRE **60**, 114 (1999)

Enhancement of observables

In a finite system, resonance trapping occurs

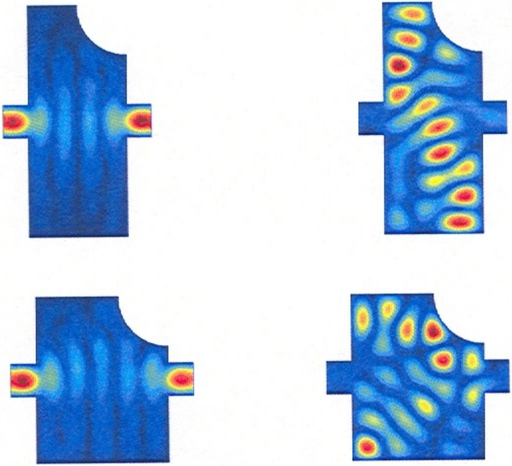
hierarchically

$\longrightarrow$ **maximal enhancement when many** $(M > K)$

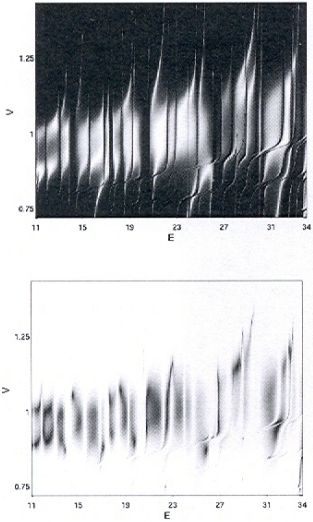
resonance states are almost aligned with

K **channel wave functions** $\leftrightarrow \ \rho \approx 0$

Numerical example: whispering gallery modes
in a Bunimovich cavity



Bulgakov, Rotter, Sadreev, 2007



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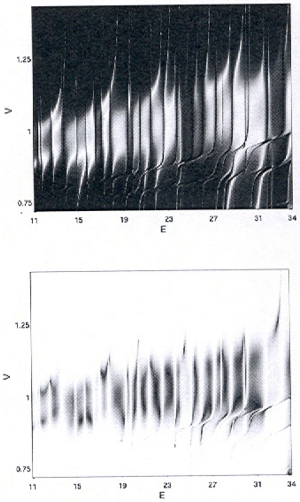

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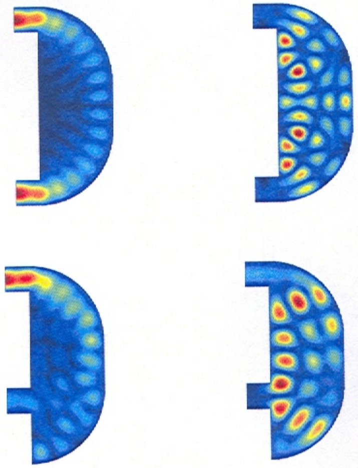

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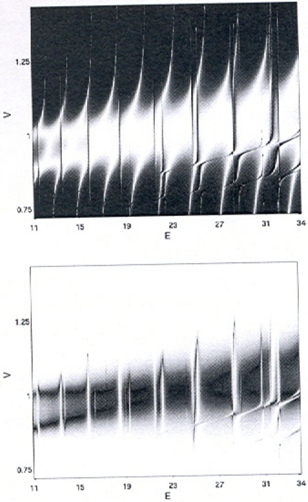

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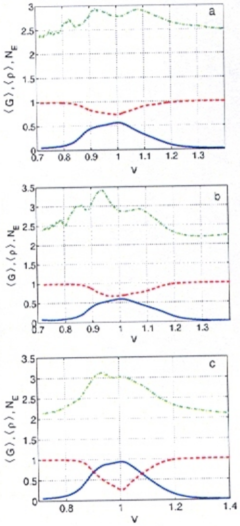

a
b
c
⟨G⟩, ⟨ρ⟩, N_E
V

Characteristic of an open quantum system in the overlapping regime

dephasing

caused by the reduction of the phase rigidity

$$r_\lambda < 1$$

of the eigenfunctions Φ_λ of the non-Hermitian Hamilton operator $H_{\rm eff}$

Summary

Mathematically, dephasing of resonance states is possible because the eigenfunctions Φ_λ of the non-Hermitian Hamilton operator H can be normalized according to

$$\langle \Phi_\lambda^{\rm left} | \Phi_\lambda^{\rm right} \rangle = \langle \Phi_\lambda^* | \Phi_\lambda \rangle = 1$$

$\Longrightarrow$ **smooth transition to the corresponding closed system with discrete eigenstates and real eigenfunctions Φ_λ^B of the Hermitian Hamilton operator H^B**

Physical consequences in the regime of overlapping resonances

- **alignment** of the scattering wave function Ψ_c^E with the channel wave function ξ_c^E
- **enhancement** of observables in the crossover from the weak-coupling regime to the strong-coupling one
- appearance of **bound states in the continuum** (caused by widths bifurcation)