$\mathcal{PT}$-Symmetric Quantum Electrodynamics—$\mathcal{PT}$QED

K. A. Milton

with C. M. Bender, I. Cavero-Peláez, P. Parashar, K.V. Shajesh, and J. Wagner

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Oklahoma Center for High Energy Physics, and

H. L. Dodge Department of Physics and Astronomy

University of Oklahoma

Transformation Properties

At the first International Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics (Prague, 2003) I proposed a $\mathcal{PT}$-symmetric version of quantum electrodynamics. A non-Hermitian but $\mathcal{PT}$-symmetric electrodynamics is based on the assumption of novel transformation properties of the electromagnetic fields under parity transformations, that is,

$$\mathcal{P}: \quad \mathbf{E} \to \mathbf{E}, \quad \mathbf{B} \to -\mathbf{B}, \quad \mathbf{A} \to \mathbf{A}, \quad A^0 \to -A^0,$$

Time Reversal

just the statement that the four-vector potential is assumed to transform as an axial vector. Under time reversal, the transformations are assumed to be conventional,

$$\mathcal{T}: \quad \mathbf{E} \rightarrow \mathbf{E}, \quad \mathbf{B} \rightarrow -\mathbf{B}, \quad \mathbf{A} \rightarrow -\mathbf{A}, \quad A^0 \rightarrow A^0.$$

Fermion fields are assumed to transform conventionally.

Lagrangian and Hamiltonian

The Lagrangian of the theory then possesses an imaginary coupling constant in order that it be invariant under the product of these two symmetries: $\mathcal{L} =$

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\psi^\dagger\gamma^0\gamma^\mu\frac{1}{i}\partial_\mu\psi+m\psi^\dagger\gamma^0\psi+ie\psi^\dagger\gamma^0\gamma^\mu\psi A_\mu.$$

The corresponding Hamiltonian density is

$$\mathcal{H}=\frac{1}{2}(E^2+B^2)+\psi^\dagger\left[\gamma^0\gamma^k\left(\frac{1}{i}\nabla_k+ieA_k\right)+m\gamma^0\right]\psi.$$

Current Density

The electric current appearing in both the Lagrangian and Hamiltonian densities, $j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi$, transforms conventionally under both $\mathcal{P}$ and $\mathcal{T}$:

$$\mathcal{P} j^\mu(\mathbf{x},t)\mathcal{P} = \begin{pmatrix} j^0 \\ -\mathbf{j} \end{pmatrix}(-\mathbf{x},t),$$

$$\mathcal{T} j^\mu(\mathbf{x},t)\mathcal{T} = \begin{pmatrix} j^0 \\ -\mathbf{j} \end{pmatrix}(\mathbf{x},-t).$$

ETCR

We are working in the Coulomb gauge, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, so the nonzero canonical equal-time commutation relations are

$$\{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = \delta_{ab}\delta(\mathbf{x} - \mathbf{y}),$$

$$[A_i^T(\mathbf{x}), E_j^T(\mathbf{y})] = -i\left[\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2}\right]\delta(\mathbf{x} - \mathbf{y}),$$

where T denotes the transverse part,

$$\boldsymbol{\nabla} \cdot \mathbf{A}^T = \boldsymbol{\nabla} \cdot \mathbf{E}^T = 0.$$

The $\mathcal{C}$ operator

As for quantum mechanical systems, and for scalar quantum field theory, we seek a $\mathcal{C}$ operator in the form

$$\mathcal{C} = e^{Q}\mathcal{P},$$

where $\mathcal{P}$ is the parity operator. $\mathcal{C}$ must satisfy the properties

$$\begin{aligned} \mathcal{C}^2 &= 1, \\ [\mathcal{C}, \mathcal{PT}] &= 0, \\ [\mathcal{C}, H] &= 0. \end{aligned}$$

Conditions on Q

From the first two equations we infer

$$Q = -\mathcal{P}Q\mathcal{P},$$

and because $\mathcal{P}\mathcal{T} = \mathcal{T}\mathcal{P}$,

$$Q = -\mathcal{T}Q\mathcal{T}.$$

Perturbative determination of Q_1

The third equation allows us to determine Q perturbatively. If we separate the interaction part of the Hamiltonian from the free part,

$$H = H_0 + eH_1,$$

and assume a perturbative expansion of Q:

$$Q = eQ_1 + e^2 Q_2 + \ldots,$$

the first contribution to the Q operator is determined by

$$[Q_1, H_0] = 2H_1.$$

Expansion of Q

The second correction commutes with the Hamiltonian,

$$[Q_2, H_0] = 0.$$

Thus we may take

$$Q = eQ_1 + e^3 Q_3 + \ldots,$$

which illustrates a virtue of the exponential form. The $O(e)$ term was explicitly computed in 2005 [Bender, Cavero-Peláez, Milton, and Shajesh, Phys. Lett. **B613** 97-104 (2005)].

$2D\ \mathcal{PT}$ QED

However, the above perturbative construction of $\mathcal{C}$ fails for 2-dimensional $\mathcal{PT}$-symmetric QED. In two dimensions, the only nonzero component of the field strength tensor is $F^{01} = E$, and the Hamiltonian of the system is $H = \int (dx)\mathcal{H}$, where the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}E^2 - iJ_1A_1 - \frac{i}{2}\psi\gamma^0\gamma^1\partial_1\psi + \frac{m}{2}\psi\gamma^0\psi,$$

where $J^\mu = \frac{1}{2}\psi\gamma^0\gamma^\mu eq\psi$. Now we're using real fields, and correspondingly an antisymmetric 2×2 charge matrix q.

Radiation Gauge

As before, we choose the radiation gauge because it is most physical:

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \partial_1 A_1 = 0,$$

and then the Maxwell equation

$$\partial_1 E_1 = -\partial_1^2 A^0 = iJ^0,$$

which implies the following construction for the scalar potential

$$A^0(x) = -\frac{i}{2}\int_{-\infty}^{\infty} dy|x-y|J^0(y).$$

Construction of E

Without loss of generality, we can disregard A_1, and then the electric field is

$$E(x) = \frac{i}{2} \int_{-\infty}^{\infty} dy \, \epsilon(x-y) J^0(y).$$

Elimination of E

Thus the electric field part of the Hamiltonian is

$$\int dx \frac{1}{2} E^2 = -\frac{1}{8} \int dx\, dy\, dz \epsilon(x-y)\epsilon(x-z) J^0(y) J^0(z)$$
$$= -\frac{1}{8} L Q^2 + \frac{1}{4} \int dy\, dz J^0(y) |y-z| J^0(z),$$

where L is the "length of space" and the total charge is

$$Q = \int dy J^0(y).$$

As this is a constant, we may disregard it.

Form of Hamiltonian

Thus we obtain the form found (for the conventional theory) years ago by Lowell Brown:

$$H = \frac{1}{4}\int dy\, dz\, J^0(y)|y-z|J^0(z)$$
$$-\int dx \left\{\frac{i}{2}\psi\gamma^0\gamma^1\partial_1\psi - \frac{m}{2}\psi\gamma^0\psi.\right\}$$

This resembles ϕ^4 theory, and for the same reason, we cannot calculate the $\mathcal{C}$ operator perturbatively. Henceforth, we will set the $m=0$.

ETCR of Currents

It is easy to check that

$$[J^0(x,t), J^0(y,t)] = 0.$$

However, it requires a point-splitting calculation to verify that

$$[J^0(x,t), J^1(y,t)] = -\frac{ie^2}{\pi}\frac{\partial}{\partial x}\delta(x-y).$$

Free Green's Function

The key element in the latter is that the singular part of the 2-point fermion correlation function is given by the free Green's function:

$$\langle \psi_\alpha(x)(\psi(y)\gamma^0)_\beta \rangle = \frac{1}{i} G_{\alpha\beta}(x-y),$$

$$G(z) = -\frac{1}{2\pi} \frac{\gamma_\mu z^\mu}{z^2 + i\epsilon}.$$

Conservation of Electric Charge

Current Conservation:

$$\partial_0 J^0 = \frac{1}{i}[J^0, H] = -\partial_1 J^1,$$

as expected from electric current conservation,

$$\partial_\mu J^\mu = 0.$$

Axial-Vector Anomaly

In 2-dimensions, the dual current is

$$ {}^*J^\mu = \epsilon^{\mu\nu} J_\nu, \quad {}^*J^0 = J_1, \quad {}^*J^1 = J^0. $$

Now, using the above commutator between J^0 and J^1, we find

$$ \partial_0 {}^*J^0 = \partial_0 J_1 = \frac{1}{i}[J_1, H] $$
$$ = -\partial_1 J^0 + \frac{1}{i}\left[J_1(x), \frac{1}{4}\int dy\, dz\, J^0(y)|y-z|J^0(z)\right] $$

Axial-Anomaly (cont.)

$$\partial_\mu{}^* J^\mu(x) = -\frac{e^2}{2\pi}\int dy\, dz \partial_x \delta(x-y)|y-z|J^0(z)$$

$$= -\frac{ie^2}{\pi}\partial_x A^0 = \frac{ie^2}{\pi}E.$$

This is the two-dimensional version of the famous Schwinger-Adler-Bell-Jackiw anomaly.

Schwinger mass generation

Combine the current conservation and axial-current non-conservation:

$$\partial_1 [\partial_0 J^0 + \partial_1 J^1 = 0]$$

$$\partial_0 \left[\partial_0 J^1 + \partial_1 J^0 = \frac{ie^2}{\pi} E \right],$$

together with the Maxwell equation

$$\partial_0 E = -iJ^1,$$

Spacelike singularity

to obtain ($\partial^2 = -\partial_0^2 + \partial_1^2$)

$$\left(\partial^2 + \frac{e^2}{\pi}\right) J^1 = 0.$$

This corresponds to a spacelike singularity, a pole at

$$p^2 = -\partial^2 = \frac{e^2}{\pi},$$

implying complex energies!

Perturbation Theory

This result is consistent with perturbation theory, where in general we expect all we have to do is replace

$$e \to ie.$$

In fact, the Schwinger mass comes from one-loop vacuum polarization. In particular, the $\mathcal{C}$ operator appears to have no effect on the weak-coupling expansion: C. M. Bender, J.-H. Chen, K. A. Milton "$\mathcal{PT}$-Symmetric Versus Hermitian Formulations of Quantum Mechanics," J. Phys. A **39** 1657 (2006).

Zero-Dimensions

We constrast the zero-dimensional partition functions for a conventional and a $\mathcal{PT}$-symmetric x^{2+N} theory.

$$Z_N^c(K) = \int_{-\infty}^{\infty} dx\, e^{-x^2 - gx^{2+N} - Kx},$$

$$Z_N(K) = \int dx\, e^{-x^2 - gx^2(ix)^N - Kx}.$$

The integral in the latter is taken in the lower half plane, so that the integrand decays exponentially fast.

Perturbation Theory

Note that the $\mathcal{PT}$-symmetric theory has a perturbation theory which doesn't appear to know about the path of integration:

$$Z_N(K) = \sqrt{\pi}\exp\left[g\left(-i\frac{d}{dK}\right)^{N+2}\right]e^{K^2/4}$$

$$= \sqrt{\pi}e^{K^2/8}\sum_{n=0}^{\infty}\left(\frac{(-1)^N g}{2^{1+N/2}}\right)^n \frac{1}{n!}D_{n(N+2)}\left(\frac{iK}{\sqrt{2}}\right).$$

$N = 2, K = 0$

For the $-x^4$ theory, we have the closed form for the vacuum amplitude

$$Z_2(0) = \frac{\pi}{4\sqrt{g}} e^{-1/8g} \left[I_{1/4}\left(\frac{1}{8g}\right) + I_{-1/4}\left(\frac{1}{8g}\right)\right].$$

Directly, or from the previous expansion, we find the weak-coupling expansion ($g \to 0$)

$$Z_2(0) \sim \sqrt{\pi}\left(1 + \frac{3}{4}g + \frac{105}{32}g^2 + \ldots\right);$$

the expansion of Z_2^c differs only in the sign of g.

Strong-coupling contrasted

Conventional theory:

$$Z_2^c(0) = \frac{1}{2\sqrt{g}} e^{1/8g} K_{1/4}\left(\frac{1}{8g}\right).$$

Even the leading terms are different: $(g \to \infty)$

$$Z_2^c(0) \sim \frac{\sqrt{2}\pi}{2g^{1/4}\Gamma(3/4)}\left[1 - \frac{1}{4\sqrt{g}}\frac{\Gamma(3/4)}{\Gamma(5/4)} + \ldots\right],$$

$$Z_2(0) \sim \frac{\pi}{2g^{1/4}\Gamma(3/4)}\left[1 + \frac{1}{4\sqrt{g}}\frac{\Gamma(3/4)}{\Gamma(5/4)} + \ldots\right].$$

Conclusions

- Perturbation theory evidently fails to give a positive spectrum to the massless $\mathcal{PT}$-symmetric electrodynamics in 2 dimensions.
- Non-perturbative effects (strong field effects) presumably resolve this issue.
- Clearly there are issues unsolved relating to fermions and gauge theories in the $\mathcal{PT}$-context.