Spectral properties of non-Hermitian systems arising in fiber optics

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- Motivation
- Physical background, the NLS
- The Zakharov-Shabat eigenvalue problem:
  - existence and location of EVs
  - EVs and shape of the potential
  - parameter dependence of EVs
  - EV collisions
  - spectral singularities

Motivation

- Pulse propagation in lossless optical fibers as described by the focusing nonlinear Schrödinger equation (joint work with Ken Shaw)

- Inverse scattering transform method

- Non-selfadjoint operators exhibit interesting eigenvalue behavior (joint work with Boris Mityagin)

Physical Background

Under certain ideal conditions optical pulses are governed by the nonlinear Schrödinger equation (focusing case)

$$iu_z = -\frac{1}{2}\,u_{tt} - \,|u|^2\,u,$$

where $u(z,t)$ is the slowly varying field envelope of the pulse (suitably normalized)

Soliton solutions:

$$u(z,t) = A\,\mathrm{sech}[A(t-\mathrm{v}z)]\,e^{i[\mathrm{v}\,t-(\mathrm{v}^2-A^2)z/2]},$$

where A is the amplitude and v is the velocity

Soliton solutions correspond to eigenvalues of a Zakharov-Shabat system (ZS)

$$v_1' = -i\,\xi\, v_1 + q(t)\, v_2$$
$$v_2' = -q(t)^*\, v_1 + i\,\xi\, v_2$$

where $q(t) = u(0,t)$ is often referred to as the potential and ξ is the spectral parameter; $q(t)^*$ is the complex conjugate of $q(t)$. We have

$$\xi = \frac{\mathrm{v}}{2} + \frac{A}{2}\, i.$$

Given $u(0,t)$ we can find $u(z,t)$ by means of the inverse scattering transform (IST).

The IST for the NLS

$$
\begin{array}{ccc}
u(0,t) & \xrightarrow{\text{forward scattering}} & SD(0) \\
\text{NLS}\big\downarrow & & \big\downarrow\text{z}-\text{evolution} \\
u(z,t) & \xleftarrow[\text{inverse scattering}]{} & SD(z)
\end{array}
$$

Scattering data $SD(z)$: eigenvalues, norming constants and a reflection coefficient

The scattering data evolve in a known way. In particular the eigenvalues are independent of ζ.

Possible application: "eigenvalue communication" (Hasegawa and Kodama: Solitons in Optical Communications, Oxford Series 1995)

The Zakharov-Shabat eigenvalue problem

We assume throughout that

$$q \in L^1(\mathbb{R}).$$

The ZS system can be written in the form

$$H\,v = (H_0 + Q)\,v = \xi\,v,$$

where

$$H_0 = iJ\frac{d}{dt}, \qquad Q = i\begin{pmatrix} 0 & -q \\ -q^* & 0 \end{pmatrix},$$

and

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

H_0 is selfadjoint, Q is skew-selfadjoint.

The spectrum may contain nonreal components.

H is (formally) J-selfadjoint,

$$JHJ = H^{\dagger}$$

(the † denotes the adjoint) and thus

$$\sigma(H) = \sigma(H)^{*}.$$

If q is even or if q is real, then the eigenvalues appear as quartets: ξ, ξ^{*}, $-\xi$, and $-\xi^{*}$.

The real axis consists of essential spectrum and there are no real eigenvalues.

Bounds on the location of eigenvalues

Theorem Suppose that $q \in L^1(\mathbb{R})$. Then:

(i) If q is of bounded variation, then the eigenvalues lie in the semi-disk

$$|\xi| \leq \frac{1}{2}\, \|q\|_1 \, \underset{\mathbb{R}}{\mathrm{Var}}\, [q].$$

(ii) If $q \in L^1 \cap L^p$, $1 < p \leq \infty$, then the eigenvalues lie in the strip

$$0 < \beta \leq \frac{\|q\|_p^r \, \|q\|_1^r}{2\, r} \quad \text{if} \quad p \neq \infty,$$

where $1/p + 1/r = 1$, $\beta = \mathrm{Im}\, \xi$.

Furthermore, if $p = \infty$, then $0 < \beta \leq \min\{\|q\|_\infty, \|q\|_\infty \, \|q\|_1/2\}$.

J. Bronski's bounds (1995):

$$\beta \leq \frac{\|q'\|_\infty}{2\, \alpha}, \qquad \beta \leq \frac{\|q''\|_\infty + \|(|q|^2)'\|_\infty}{\alpha^2} \quad (\alpha = \mathrm{Re}\, \xi).$$

Shape-dependence of EVs

A real potential in $L^1(\mathbb{R})$ is called **single lobe** if there is a value t_0 such that $q(t)$ is nondecreasing for $t < t_0$ and nonincreasing for $t > t_0$.

Theorem (2002, with Ken Shaw) Suppose q (or -q) is single lobe. Then there are no non-imaginary eigenvalues and every imaginary eigenvalue has (algebraic) multiplicity 1.

Recent extension (with B. Mityagin): If q is even, positive on $(-d, d)$, absolutely continuous on $[-d, d]$, $q(t) = 0$ for $|t| > d$, and if q has a single maximum at the point t_0 $(0 < t_0 \leq d)$ such that $q(t_0) < 2\,q(0)$, then all eigenvalues are purely imaginary.

A connection with PT-symmetric Hamiltonians

Suppose that q is real and smooth. The ZS eigenvalue problem reads

$$H\,v = (H_0 + Q)\,v = \xi\,v$$

where

$$H_0 = iJ\frac{d}{dt}, \quad Q = i\begin{pmatrix} 0 & -q \\ -q & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Squaring gives

$$(H_0{+}Q)^2\,v = \xi^2\,v \Longleftrightarrow \begin{pmatrix} A & 0 \\ 0 & A^\dagger \end{pmatrix} w = \xi^2\,w,$$

where

$$A = -d^2/dt^2 + iq' - q^2.$$

If q is also even, then A and $A^\dagger$ are PT-symmetric. Then we have a “single lobe result” for A, which says that A can only have real eigenvalues.

What happens if the potential is not single lobe?

We consider the ZS system with potential

$$q(t) = \mu\, p(t),$$

where $\mu \geq 0$ is a coupling constant.

Thus the EVs become functions of μ. We are particularly interested in those values of μ for which

- two EVs collide
- an EV branch $\xi(\mu)$ enters or leaves $\mathbb{C}^+$ at some point ξ on the real axis (spectral singularities)

We assume from now on that p is real and even and that p has compact support $[-d, d]$.

Eigenvalue collisions

Every eigenfunction belongs to one of two symmetry subspaces:

$$\mathcal{S}_{\pm} = \{v(t) \text{ bounded} : v_1(0) = \pm v_2(0)\}$$

We note that $\xi(\mu)$ is determined by the equation

$$v_1(d; \xi(\mu), \mu) = 0$$

and that we may assume

$$v_1(-d; \xi, \mu) = 1, v_2(-d; \xi, \mu) = 0.$$

A useful tool is the formula

$$\xi'(\mu) =$$

$$-\frac{i \int_{-d}^{d} p(t) \left[v_1(t; \xi(\mu), \mu)^2 + v_2(t; \xi(\mu), \mu)^2\right] dt}{2 \int_{-d}^{d} v_1(t; \xi(\mu), \mu) v_2(t; \xi(\mu), \mu) dt}$$

Since at $\xi = 0$ we have

$$v_1(t;0,\mu) = \cos\left(\mu \int_{-d}^{t} p(\tau)d\tau\right)$$

$$v_2(t;0,\mu) = -\sin\left(\mu \int_{-d}^{t} p(\tau)d\tau\right)$$

it follows that

$$v_1(d;0,\mu) = 0 \Longleftrightarrow \mu = \mu_k = \frac{(2k-1)\pi}{4\int_0^d p(t)dt}$$

for $k = 1, 2, 3, \ldots$.

Then

$$\xi'(\mu_k) = \frac{i(-1)^{k+1}\int_0^d p(t)dt}{\int_0^d \cos(2\mu_k \int_0^t p(\tau)d\tau)dt}$$

whenever $\xi(\mu)$ is an EV branch such that $\xi(\mu_k) = 0$.

Therefore

$\mathrm{Im}\,\xi'(\mu_k) > 0 \Longrightarrow$ an EV appears as $\mu \uparrow \mu_k + \epsilon$
$\mathrm{Im}\,\xi'(\mu_k) < 0 \Longrightarrow$ an EV is absorbed as $\mu \downarrow \mu_k$

If k is odd (even), then $v(t;0,\mu_k) \in \mathcal{S}_-(\mathcal{S}_+)$.

Lemma Suppose $v_1(d;\xi(\mu_k),\mu_k)=0$ and $\operatorname{Im}\xi'(\mu_k)<0$ for some k. Then there exists $0<\mu_k^c<\mu_k$ such that $\xi(\mu)$ is a simple purely imaginary EV for $\mu_k^c<\mu<\mu_k$ and a non-simple EV for $\mu=\mu_k^c$.

This means that the EV must originate from a collision of two (or more) EVs.

Hypothesis (H) Suppose that p has support $[-d,d]$, is positive, even, and has the property that there exists an $a\in(0,d)$ such that p is absolutely continuous on each of the subintervals $[0,a)$ and $(a,d]$.

Let

$$N_{tot}(\mu)=\#\{k:0<\mu_k\leq\mu\},$$
$$N_{up}(\mu)=\#\{k:0<\mu_k\leq\mu \text{ and } \operatorname{Im}\xi'(\mu_k)>0\},$$
$$N_{down}(\mu)=\#\{k:0<\mu_k\leq\mu \text{ and } \operatorname{Im}\xi'(\mu_k)<0\},$$

Let

$$\omega=\frac{\int_0^a p(t)\,dt}{2\int_0^d p(t)\,dt},\qquad \rho=\frac{p(a+)\,p(a-)}{p(d)\,(p(a+)-p(a-))}.$$

Theorem Assume hypothesis (H) holds. Then the following are true:

(*i*) If $|\rho| > 1$, then Im $\xi'(\mu_k) > 0$ for all sufficiently large k.

(*ii*) If $|\rho| \leq 1$ and $\omega \notin \mathbb{Q}$, then

$$\lim_{\mu \to \infty} \frac{N_{down}(\mu)}{N_{tot}(\mu)} = \frac{1}{2} - \frac{\arcsin |\rho|}{\pi}$$

We also have results when $|\rho| \leq 1$ and $\omega \in \mathbb{Q}$.

The theorem says that in case (*ii*) infinitely many collisions must occur.

Location of collisions

Theorem Suppose

(i) $p(t)$ is even, $p(t) = 0$ for $0 \leq t < a$,

(ii) on $[a, d]$, $p(t)$ is positive and $p'(t) \in L^\infty$.

Then there exist positive constants c_1, c_2 such that if $\xi(\mu)$ is a non-simple, purely imaginary eigenvalue of the ZS system on the subspace $\mathcal{S}_+$, then $\xi(\mu) \in (c_1 \ln \mu, c_2 \ln \mu)$ provided μ is sufficiently large. As μ increases, eigenvalues in $(0, c_1 \ln \mu]$ move downward and eigenvalues in $[c_2 \ln \mu, \infty)$ move upward.

There are no non-simple imaginary EVs on $\mathcal{S}_-$ provided μ is sufficiently large.

Let

$$\widehat{\mu}_k = \frac{(2k-1)\pi}{2\int_a^d p(t)\,dt},$$

$$\omega_1 = \int_a^d p(t)\,dt, \qquad \omega_2 = \frac{1}{2}\int_a^d \frac{dt}{p(t)}.$$

Theorem Suppose that $p(t)$ is even, $p(t) = 0$ for $0 \le t < a$, positive and absolutely continuous for $a \le t \le d$, and zero for $t > d$.

Then collisions on $\mathcal{S}_+$ occur for

$$\mu_k^c = \widehat{\mu}_k + \frac{\omega_2}{4\,a^2\,\omega_1}\,\frac{[\ln(2a\widehat{\mu}_k)]^2}{\widehat{\mu}_k} + O\left(\frac{(\ln\widehat{\mu}_k)\,\ln(\ln\widehat{\mu}_k)}{\widehat{\mu}_k}\right),$$

at $\xi = i\,s_k^c$, where

$$s_k^c = \frac{\ln(2a\widehat{\mu}_k)}{2a} - \frac{1}{2a}\ln\left(\frac{\omega_2}{a}\,\ln(2a\widehat{\mu}_k)\right) + O\left(\frac{\ln(\ln\widehat{\mu}_k)}{\ln\widehat{\mu}_k}\right).$$

Spectral singularities

Spectral singularities are points $\xi \in \mathbb{R}$ where $v_1(d; \xi, \mu) = 0$.

A typical result There exist two pairs of sequences (ξ_k^-, μ_k^-) and (ξ_k^+, μ_k^+) such that

$$\xi_k^- \uparrow \frac{\pi}{2a}, \qquad \mu_k^- \to \infty$$

$$\xi_k^+ \downarrow \frac{\pi}{2a}, \qquad \mu_k^+ \to \infty$$

and $v_1(d; \xi_k^-, \mu_k^-) = 0$, $v_1(d; \xi_k^+, \mu_k^+) = 0$.

Similar results hold at $\frac{n\pi}{2a}$, $n = 2, 3, \ldots$.

Asymptotics:

$$\xi_k^- = \frac{\pi}{2a} - \frac{\int_a^d p(t)\,dt}{4a^2 p(d) k} + O\left(\frac{1}{k^2}\right)$$

$$\mu_k^- = \frac{k\pi}{\int_a^d p(t)\,dt} + \frac{\pi}{4\int_a^d p(t)\,dt} + O\left(\frac{1}{k}\right)$$