PT Symmetry, Continuous Symmetries, and Large-N

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Motivation

- Importance of continuous symmetries
  - Conservation laws
  - Gauge symmetries
- Importance of Large-N limit
  - Powerful insights
  - Many different approaches
  - Useful in many areas of physics

Topics

- Matrix Models with U(N) Symmetry
  - hep-th/0701207
- Vector Models with O(N) Symmetry
  - arXiv:0707.1655 [hep-th]

Matrix Model Formalism

Hermitian matrix models — Brezin *et al.* (1978)

$$\frac{1}{2}Tr\left(\frac{dM}{dt}\right)^2 + \frac{g}{N}TrM^4$$

More generally

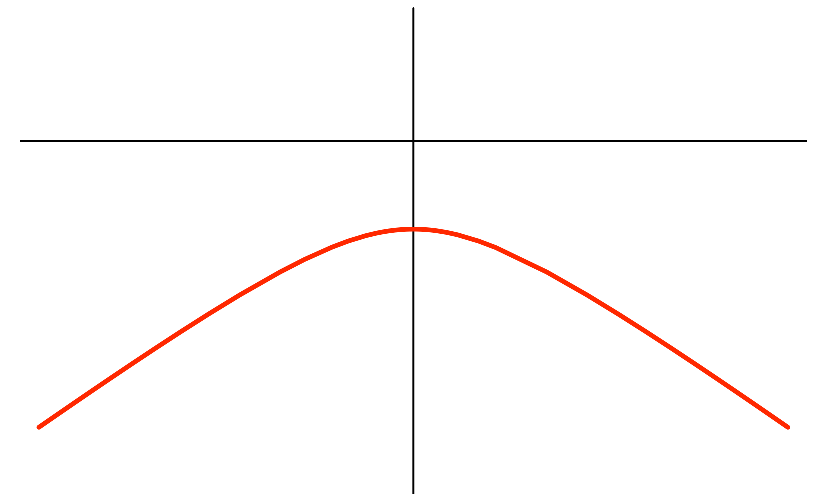
$$L = \frac{1}{2}Tr\left(\frac{dM}{dt}\right)^2 + TrV(M)$$

Note that the potential depends only on the eigenvalues of M:

$$Tr\,V(M) = \sum_j V(\lambda_j) \qquad M = U\Lambda U^+$$

PT-Symmetric Matrix Models

$$L = \frac{1}{2} Tr \left(\frac{dM}{dt} \right)^2 - \frac{g}{N^{p/2-1}} Tr \left(iM \right)^p$$



- Typical, rather than general case
- Defined by extending M from Hermitian to normal matrices: eigenvalues λ_j become complex.
- Assume that ground state is a singlet for all p.
  - Can prove for p=2,4
  - Would be nice to have general proof!

Ground State Wave Function

- Singlet wave functions ψ are symmetric functions of eigenvalues.
- Tranformation of the wave function plus separation of variables reduce the equation to a single component.

$$\phi\left(\lambda_1, .., \lambda_N\right) = \left[\prod_{j<k}\left(\lambda_j - \lambda_k\right)\right]\psi\left(\lambda_1, .., \lambda_N\right)$$

$$H = \frac{1}{2}p^2 - \frac{g}{N^{p/2-1}}\left(i\lambda\right)^p \qquad \phi\left(\lambda_1, .., \lambda_N\right) = \prod_j \phi_{k_j}\left(\lambda_j\right)$$

Problem becomes fermionic!

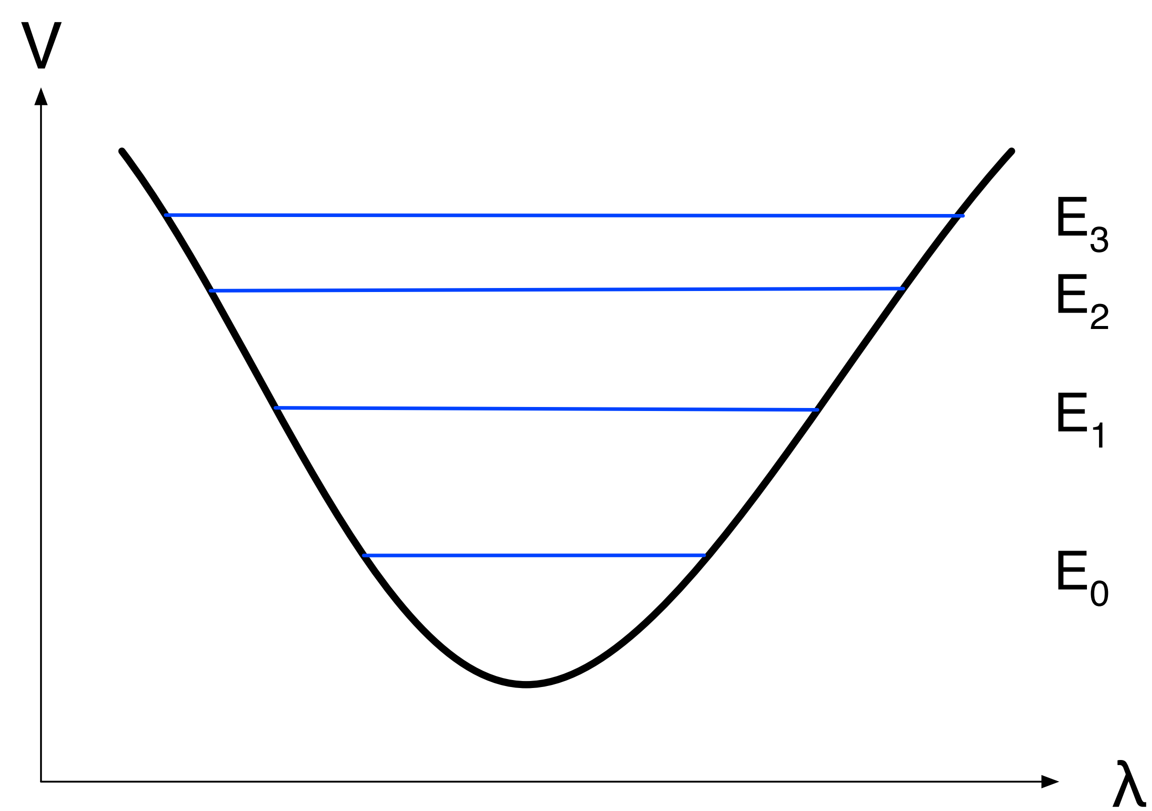
Symmetric ψ gives rise to antisymmetric Φ

$$\phi\left(\lambda_1,..,\lambda_N\right)=\left[\prod_{j<k}\left(\lambda_j-\lambda_k\right)\right]\psi\left(\lambda_1,..,\lambda_N\right)$$

Pauli exclusion principle: in ground state, fill lowest N levels

$$\phi\left(\lambda_1,..,\lambda_N\right)=\prod_j \phi_{k_j}\left(\lambda_j\right)$$

$$H=\frac{1}{2}p^2-\frac{g}{N^{p/2-1}}\left(i\lambda\right)^p$$



WKB in the Complex Plane for Large-N

$$N = \frac{1}{2\pi}\int dp d\lambda\ \theta\left[E_F - H(p,\lambda)\right]$$

- Formula must be interpreted by integrating over p first.
- Formula sets top of Fermi sea.
- Once Fermi energy is known, total ground energy is sum of individual enegies up to Fermi level.

$$E_\infty^{(0)} = \frac{1}{2\pi}\int dp d\lambda\, H_{sc}(p,\lambda)\theta\left[\epsilon_F - H_{sc}(p,\lambda)\right]$$

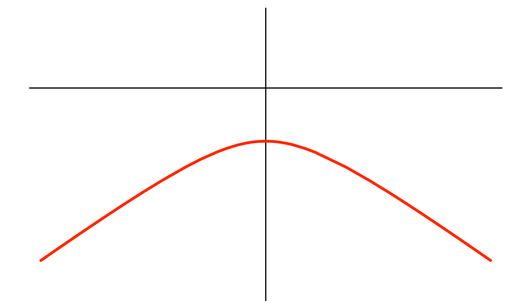
Results for p=3,4

$$E_{\infty}^{(0)} = \frac{p+2}{3p+2}\left[\left(\frac{\pi}{2}\right)^p\left(\frac{\Gamma(3/2+1/p)}{\sin(\pi/p)\,\Gamma(1+1/p)}\right)^{2p} g^2\right]^{\frac{1}{p+2}}$$

Integral can be carried out using the same two-segment path used in Bender and Boettcher (1998): straight-line paths from the complex classical turning points to the origin.

N	p=3	p=4
1	0.762852	0.930546
2	0.756058	0.935067
3	0.754860	0.935846
4	0.754443	0.936115
5	0.754251	0.936239
6	0.754147	0.936306
7	0.754084	0.936347
8	0.754043	0.936372
∞	0.753991	0.936458

Special Case: Tr M^4



PT-symmetric and Hermitian models are isospectral!

$$M = -2i\sqrt{1+iH}$$

$$L_{PT} = \frac{1}{2}Tr\left(\frac{dM}{dt}\right)^2 + \frac{1}{2}m^2 Tr\,M^2 - \frac{g}{N}TrM^4$$

$$L_H = \frac{1}{2}Tr\left(\frac{d\Pi}{dt}\right)^2 - \sqrt{\frac{2g}{N}}Tr\Pi - m^2 Tr\Pi^2 + \frac{4g}{N}Tr\Pi^4$$

- Proof follows Jones *et al.* (2006).
- Many features of N=1 case repeat.
- Anomaly dissappears in the large-N limit.
- Includes non-singlet states!
- Singlet nature of ground state follows.

Conclusions for Matrix Models

- The extension from Hermitian matrix models to PT-symmetric models is straightforward.
- The large-N limit can be constructed, and treated via WKB in a manner similar to Hermitian models.
- Numerical results show a rapid approach to the large-N limit as N increases.
- The PT-matrix anharmonic oscillator is equivalent to a Hermitian matrix anharmonic oscillator, generalizing one-component results.

PT-Symmetric Models with O(N) Symmetry

$$L_E = \sum_{j=1}^{N} \left[\frac{1}{2} (\partial_t x_j)^2 + \frac{1}{2} m^2 x_j^2 \right] - \frac{g}{N} \left(\sum_{j=1}^{N} x_j^2 \right)^2$$

- Obvious O(N) symmetry.
- Mixed problem: radial mode plus angular modes involve different physics.
- Change of variable for single-component case is problematic here.

An Extended Model

$$L_E = \sum_{j=1}^{N} \left[\frac{1}{2} \left(\partial_t x_j \right)^2 + \frac{1}{2} m^2 x_j^2 - \lambda x_j^4 \right] - \frac{g}{N} \left(\sum_{j=1}^{N} x_j^2 \right)^2$$

- g=0: N single-component PT-symmetric models
- λ=0: O(N) symmetric model
- Interaction defined by analytic continuation from p,q=1

$$-\lambda \sum_{j=1}^{N} \left(-i x_j \right)^{2p} - \frac{g}{N} \left(-\sum_{j=1}^{N} x_j^2 \right)^q$$

An Even More General Model

A general quartic interaction:

$$L_E = \sum_{j=1}^{N} \left[\frac{1}{2} (\partial_t x_j)^2 + \frac{1}{2} m^2 x_j^2 \right] - \sum_{j,k=1}^{N} x_j^2 \Lambda_{jk} x_k^2$$

For our case: $\Lambda = \lambda I + gP$

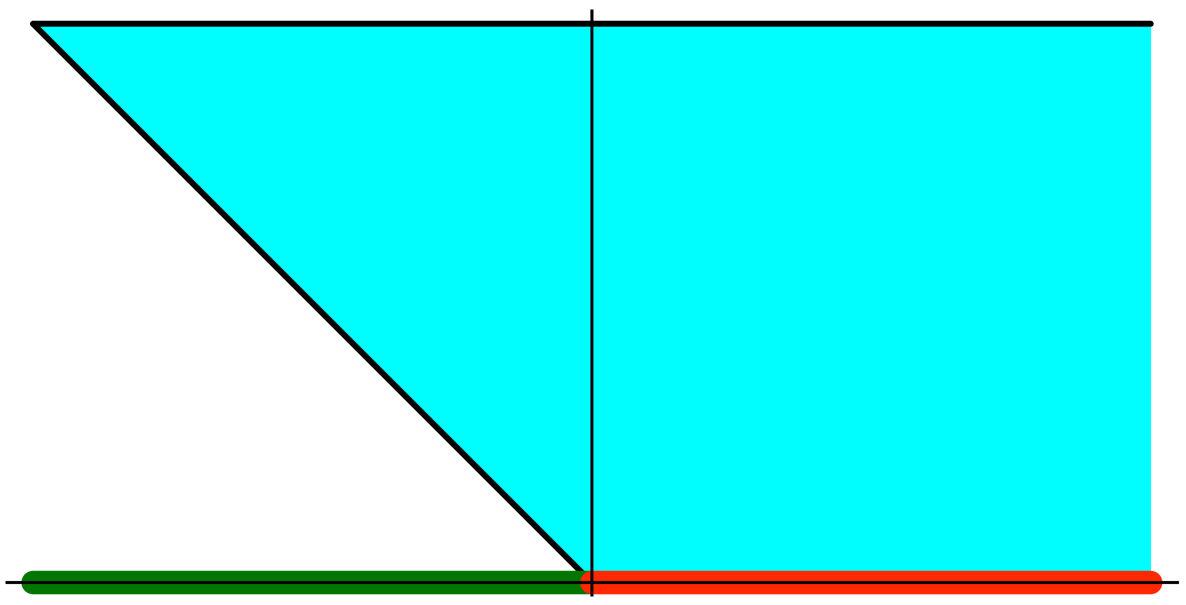
P is the projector for the radial mode:

$$P = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & .. \\ 1 & .. & .. \end{pmatrix} \quad P^2 = P$$

Strategy

$$\Lambda = (I - P) + (g + \lambda)\, P$$

- Λ has 1 eigenvalue g+λ and N-1 eigenvalues λ.
- PT- model is defined over an extended region.
- PT-symmetric model is recovered in the limit λ goes to zero.



Equivalence to Hermitian Models

$$L_E = \sum_j \left[\frac{1}{2}\dot{h}_j^2 + 4\lambda \left(h_j^2 - \frac{m^2}{8\lambda} \right)^2 \right] - \sqrt{2\lambda} \sum_j h_j$$

$$- \frac{g}{N\left(g+\lambda\right)} \left[\frac{1}{2} \left(\sum_j \dot{h}_j \right)^2 + 4\lambda \left(\sum_j \left(h_j^2 - \frac{m^2}{8\lambda} \right) \right)^2 \right]$$

- Proof again follows Jones *et al.* (2006)
  - ✦ Substitution, with inclusion of ΔV
  - ✦ Promotion of functional determinant into action
  - ✦ Integration by parts
  - ✦ Functional integration over the original fields
- Many features of N=1 case repeat.
  - ✓ Quartic interactions
  - ✓ Mass term flips sign
  - ✓ Anomaly present

The case N=2

- Thus far we have complete permutation symmetry.
- Taking λ to zero requires a rescaling that breaks that symmetry.
- Even before taking the limit, there is a natural scale separating π and σ

$$h_1 = \frac{1}{\sqrt{2}}(\sigma + \pi)$$
$$h_2 = \frac{1}{\sqrt{2}}(\sigma - \pi)$$

$$\sigma \to \sqrt{\frac{g+\lambda}{\lambda}}\sigma$$

$$L_E = \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}\dot{\pi}^2 - m^2\sigma^2 - \frac{\lambda m^2}{g+\lambda}\pi^2 + 2(g+\lambda)\sigma^4 + \frac{2\lambda^2}{g+\lambda}\pi^4 + (8g+12\lambda)\sigma^2\pi^2 - 2\sqrt{g+\lambda}\sigma$$

N=2 and the Limit λ=0

$$L_E = \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}\dot{\pi}^2 - m^2\sigma^2 + 2g\sigma^4 + 8g\sigma^2\pi^2 - 2\sqrt{g}\sigma$$

- No obvious O(2) invariance.
- π has no mass term, and no quartic self coupling. π appears only quadratically.
- Anomaly involves only sigma, and breaks σ's discrete symmetry.
- Interactions with σ will give π a mass.

The Hermitian Form of the O(N) Model

$$L_E = \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}\dot{\vec{\pi}}^2 - m^2\sigma^2 + \frac{4g}{N}\sigma^4 + \frac{16g}{N}\sigma^2\vec{\pi}^2 - \sqrt{2gN}\sigma$$

- Has the same features the N=2 case has.
- O(N-1) symmetry manifest.

Could we have guessed this?

The Large-N Limit of the O(N) Model

- We can take the large-N limit by rescaling σ.
- Integrating over the (N-1) π fields gives the large-N effective potential.

$$\sigma \to \sqrt{N}\sigma$$

$$L_E = \frac{N}{2}\dot{\sigma}^2 + \frac{1}{2}\dot{\vec{\pi}}^2 - Nm^2\sigma^2 + 4gN\sigma^4 + 16g\sigma^2\vec{\pi}^2 - N\sqrt{2g}\sigma$$

$$V_{eff}/N = -m^2\sigma^2 + 4g\sigma^4 + \frac{1}{2}\sqrt{32g\sigma^2} - \sqrt{2g}\sigma$$

An “Alternate Derivation”

Let’s return to the orginal O(N) invariant Lagrangian:

$$L_E = \sum_{j=1}^{N} \left[\frac{1}{2} \left(\partial_t x_j\right)^2 + \frac{1}{2} m^2 x_j^2 \right] - \frac{g}{N} \left(\sum_{j=1}^{N} x_j^2 \right)^2$$

We introduce a constraint field ρ

Coleman et al. (1974)

$$L_E \rightarrow L_E + \frac{g}{N} \left(\frac{2N\rho}{g} + \sum_{j=1}^{N} x_j^2 - \frac{Nm^2}{4g} \right)^2$$

$$L_E = \sum_{j=1}^{N} \left[\frac{1}{2} \left(\partial_t x_j\right)^2 + 4\rho x_j^2 \right] + \frac{4N\rho^2}{g} - \frac{Nm^2\rho}{g} + \frac{Nm^4}{16g}$$

Original fields now quadratic.

Back to the Future?

- If we integrate over the x's in the most naive and unjustified way, we obtain the large-N effective potential for ρ.
- It is essentially identical to our previous result.

$$V_{eff}/N = \frac{4\rho^2}{g} - \frac{m^2\rho}{g} + \sqrt{2\rho} + \frac{m^4}{16g}$$

$$\rho = g\sigma^2$$

$$V_{eff}/N = -m^2\sigma^2 + 4g\sigma^4 + \frac{1}{2}\sqrt{32g\sigma^2} - \sqrt{2g}\sigma$$

PT-Symmetric Field Theories

- If we boldly extend this reasoning to field theory....

$$-g\left(\vec{\phi}^2\right)^2$$

$$V_{eff}/N = \frac{4\rho^2}{g} - \frac{m^2\rho}{g} + \frac{m^4}{16g} + \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \ln\left[k^2 + 8\rho\right]$$

- Of course, this gives asymptotic freedom in the large-N limit..

$$\beta = -g^2/2\pi^2$$

Conclusions for O(N) Models

- We now know the Hermitian form of O(N)-invariant PT-symmetric quantum mechanics. Its form has many unusual features.
- Its large-N limit can be derived in a simple way that we cannot yet justify.
- Arguments suggest that PT-symmetric field theories are asymptotically free, at least in the large-N limit.