PT Symmetry, Continuous Symmetries, and Large-N

Michael Ogilvie
Washington University
St. Louis, MO 63130 USA
mco@physics.wustl.edu
Motivation

• Importance of continuous symmetries
  - Conservation laws
  - Gauge symmetries

• Importance of Large-N limit
  - Powerful insights
  - Many different approaches
  - Useful in many areas of physics
Topics

- Matrix Models with U(N) Symmetry
  - hep-th/0701207
- Vector Models with O(N) Symmetry
Matrix Model Formalism

Hermitian matrix models

\[ L = \frac{1}{2} Tr \left( \frac{dM}{dt} \right)^2 + \frac{g}{N} Tr M^4 \]

More generally

\[ L = \frac{1}{2} Tr \left( \frac{dM}{dt} \right)^2 + Tr V(M) \]

Note that the potential depends only on the eigenvalues of \( M \):

\[ Tr V(M) = \sum_{j} V(\lambda_j) \quad M = U \Lambda U^+ \]
PT-Symmetric Matrix Models

\[ L = \frac{1}{2} \text{Tr} \left( \frac{dM}{dt} \right)^2 - \frac{g}{N^{p/2-1}} \text{Tr} \left( iM \right)^p \]

- Typical, rather than general case
- Defined by extending $M$ from Hermitian to normal matrices: eigenvalues $\lambda_j$ become complex.
- Assume that ground state is a singlet for all $p$.
  - Can prove for $p=2,4$
  - Would be nice to have general proof!
Ground State Wave Function

• Singlet wave functions $\Psi$ are symmetric functions of eigenvalues.

• Transformation of the wave function plus separation of variables reduce the equation to a single component.

\[
\phi (\lambda_1, .., \lambda_N) = \left[ \prod_{j < k} (\lambda_j - \lambda_k) \right] \psi (\lambda_1, .., \lambda_N)
\]

\[
H = \frac{1}{2} p^2 - \frac{g}{N^{p/2-1}} (i\lambda)^p
\]

\[
\phi (\lambda_1, .., \lambda_N) = \prod_j \phi_{k,j} (\lambda_j)
\]
Problem becomes fermionic!

Symmetric $\psi$ gives rise to antisymmetric $\Phi$

$$\phi (\lambda_1, .., \lambda_N) = \prod_{j < k} (\lambda_j - \lambda_k) \psi (\lambda_1, .., \lambda_N)$$

Pauli exclusion principle:
in ground state, fill lowest $N$ levels

$$\phi (\lambda_1, .., \lambda_N) = \prod_j \phi_{k_j} (\lambda_j)$$

$$H = \frac{1}{2} p^2 - \frac{g}{Np/2-1} (i\lambda)^p$$
WKB in the Complex Plane for Large-N

\[
N = \frac{1}{2\pi} \int dpd\lambda \theta [E_F - H(p, \lambda)]
\]

- Formula must be interpreted by integrating over \( p \) first.
- Formula sets top of Fermi sea.
- Once Fermi energy is known, total ground energy is sum of individual energies up to Fermi level.

\[
E^{(0)}_{\infty} = \frac{1}{2\pi} \int dpd\lambda H_{sc}(p, \lambda) \theta [\epsilon_F - H_{sc}(p, \lambda)]
\]
Results for $p=3,4$

$$E^{(0)}_\infty = \frac{p + 2}{3p + 2} \left[ \left( \frac{\pi}{2} \right)^p \left( \frac{\Gamma(3/2 + 1/p)}{\sin(\pi/p) \Gamma(1 + 1/p)} \right)^{2p} g^2 \right]^{\frac{1}{p+2}}$$

<table>
<thead>
<tr>
<th>N</th>
<th>$p=3$</th>
<th>$p=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.762852</td>
<td>0.930546</td>
</tr>
<tr>
<td>2</td>
<td>0.756058</td>
<td>0.935067</td>
</tr>
<tr>
<td>3</td>
<td>0.754860</td>
<td>0.935846</td>
</tr>
<tr>
<td>4</td>
<td>0.754443</td>
<td>0.936115</td>
</tr>
<tr>
<td>5</td>
<td>0.754251</td>
<td>0.936239</td>
</tr>
<tr>
<td>6</td>
<td>0.754147</td>
<td>0.936306</td>
</tr>
<tr>
<td>7</td>
<td>0.754084</td>
<td>0.936347</td>
</tr>
<tr>
<td>8</td>
<td>0.754043</td>
<td>0.936372</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.753991</td>
<td>0.936458</td>
</tr>
</tbody>
</table>

Integral can be carried out using the same two-segment path used in Bender and Boettcher (1998): straight-line paths from the complex classical turning points to the origin.
Special Case: Tr $M^4$

PT-symmetric and Hermitian models are isospectral!

$$M = -2i\sqrt{1 + iH}$$

$$L_{PT} = \frac{1}{2} \text{Tr} \left( \frac{dM}{dt} \right)^2 + \frac{1}{2} m^2 \text{Tr} M^2 - \frac{g}{N} \text{Tr} M^4$$

$$L_H = \frac{1}{2} \text{Tr} \left( \frac{d\Pi}{dt} \right)^2 - \sqrt{\frac{2g}{N}} \text{Tr} \Pi - m^2 \text{Tr} \Pi^2 + \frac{4g}{N} \text{Tr} \Pi^4$$

- Proof follows Jones et al. (2006).
- Many features of $N=1$ case repeat.
- Anomaly dissappears in the large-$N$ limit.
- Includes non-singlet states!
- Singlet nature of ground state follows.
Conclusions for Matrix Models

• The extension from Hermitian matrix models to PT-symmetric models is straightforward.
• The large-N limit can be constructed, and treated via WKB in a manner similar to Hermitian models.
• Numerical results show a rapid approach to the large-N limit as N increases.
• The PT-matrix anharmonic oscillator is equivalent to a Hermitian matrix anharmonic oscillator, generalizing one-component results.
PT-Symmetric Models with O(N) Symmetry

\[ L_E = \sum_{j=1}^{N} \left[ \frac{1}{2} (\partial_t x_j)^2 + \frac{1}{2} m^2 x_j^2 \right] - \frac{g}{N} \left( \sum_{j=1}^{N} x_j^2 \right)^2 \]

- Obvious O(N) symmetry.
- Mixed problem: radial mode plus angular modes involve different physics.
- Change of variable for single-component case is problematic here.
An Extended Model

\[ L_E = \sum_{j=1}^{N} \left[ \frac{1}{2} (\partial_t x_j)^2 + \frac{1}{2} m^2 x_j^2 - \lambda x_j^4 \right] - \frac{g}{N} \left( \sum_{j=1}^{N} x_j^2 \right)^2 \]

- \( g=0 \): \( N \) single-component PT-symmetric models
- \( \lambda=0 \): \( O(N) \) symmetric model
- Interaction defined by analytic continuation from \( p,q=1 \)

\[-\lambda \sum_{j=1}^{N} (-ix_j)^{2p} - \frac{g}{N} \left( -\sum_{j=1}^{N} x_j^2 \right)^q\]
An Even More General Model

A general quartic interaction:

\[ L_E = \sum_{j=1}^{N} \left[ \frac{1}{2} (\partial_t x_j)^2 + \frac{1}{2} m^2 x_j^2 \right] - \sum_{j,k=1}^{N} x_j^2 \Lambda_{jk} x_k^2 \]

For our case: \( \Lambda = \lambda I + gP \)

\( P \) is the projector for the radial mode:

\[ P = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & .. \\ 1 & .. & .. \end{pmatrix} \quad P^2 = P \]
Strategy

• $\Lambda$ has 1 eigenvalue $g + \lambda$ and $N-1$ eigenvalues $\lambda$.
• PT-model is defined over an extended region.
• PT-symmetric model is recovered in the limit $\lambda$ goes to zero.

\[ \Lambda = (I - P) + (g + \lambda) P \]
Equivalence to Hermitian Models

\[ L_E = \sum_j \left[ \frac{1}{2} \dot{h}_j^2 + 4\lambda \left( h_j^2 - \frac{m^2}{8\lambda} \right)^2 \right] - \sqrt{2}\lambda \sum_j h_j \]

\[- \frac{g}{N(g + \lambda)} \left[ \frac{1}{2} \left( \sum_j \dot{h}_j \right)^2 + 4\lambda \left( \sum_j \left( h_j^2 - \frac{m^2}{8\lambda} \right) \right)^2 \right]\]

- Proof again follows Jones et al. (2006)
  - Substitution, with inclusion of \( \Delta V \)
  - Promotion of functional determinant into action
  - Integration by parts
  - Functional integration over the original fields
- Many features of N=1 case repeat.
  - Quartic interactions
  - Mass term flips sign
  - Anomaly present
The case $N=2$

- Thus far we have complete permutation symmetry.
- Taking $\lambda$ to zero requires a rescaling that breaks that symmetry.
- Even before taking the limit, there is a natural scale separating $\pi$ and $\sigma$

$$h_1 = \frac{1}{\sqrt{2}} (\sigma + \pi)$$
$$h_2 = \frac{1}{\sqrt{2}} (\sigma - \pi)$$

$$\sigma \rightarrow \sqrt{\frac{g + \lambda}{\lambda}} \sigma$$

$$L_E = \frac{1}{2} \dot{\sigma}^2 + \frac{1}{2} \dot{\pi}^2 - m^2 \sigma^2 - \frac{\lambda m^2}{g + \lambda} \pi^2 + 2 (g + \lambda) \sigma^4 + \frac{2 \lambda^2}{g + \lambda} \pi^4 + (8g + 12\lambda) \sigma^2 \pi^2 - 2 \sqrt{g + \lambda} \sigma$$
$N=2$ and the Limit $\lambda=0$

$$L_E = \frac{1}{2} \dot{\sigma}^2 + \frac{1}{2} \dot{\pi}^2 - m^2 \sigma^2 + 2g\sigma^4 + 8g\sigma^2\pi^2 - 2\sqrt{g}\sigma$$

- No obvious $O(2)$ invariance.
- $\pi$ has no mass term, and no quartic self coupling. $\pi$ appears only quadratically.
- Anomaly involves only sigma, and breaks $\sigma$’s discrete symmetry.
- Interactions with $\sigma$ will give $\pi$ a mass.
The Hermitian Form of the $\text{O}(N)$ Model

$$L_E = \frac{1}{2} \dot{\sigma}^2 + \frac{1}{2} \dot{\vec{\pi}}^2 - m^2 \sigma^2 + \frac{4g}{N} \sigma^4 + \frac{16g}{N} \sigma^2 \vec{\pi}^2 - \sqrt{2gN} \sigma$$

- Has the same features the $N=2$ case has.
- $\text{O}(N-1)$ symmetry manifest.

Could we have guessed this?
The Large-N Limit of the O(N) Model

- We can take the large-N limit by rescaling $\sigma$. 
- Integrating over the $(N-1)$ $\pi$ fields gives the large-N effective potential.

$$L_E = \frac{N}{2} \dot{\sigma}^2 + \frac{1}{2} \dot{\pi}^2 - N m^2 \sigma^2 + 4gN \sigma^4 + 16g\sigma^2 \pi^2 - N \sqrt{2g}\sigma$$

$$\frac{V_{eff}}{N} = -m^2 \sigma^2 + 4g\sigma^4 + \frac{1}{2} \sqrt{32g}\sigma^2 - \sqrt{2g}\sigma$$
An “Alternate Derivation”

Let’s return to the original $O(N)$ invariant Lagrangian:

$$L_E = \sum_{j=1}^{N} \left[ \frac{1}{2} \left( \partial_t x_j \right)^2 + \frac{1}{2} m^2 x_j^2 \right] - \frac{g}{N} \left( \sum_{j=1}^{N} x_j^2 \right)^2$$

We introduce a constraint field $\rho$

$$L_E \to L_E + \frac{g}{N} \left( \frac{2N \rho}{g} + \sum_{j=1}^{N} x_j^2 - \frac{Nm^2}{4g} \right)^2$$

$$L_E = \sum_{j=1}^{N} \left[ \frac{1}{2} \left( \partial_t x_j \right)^2 + 4\rho x_j^2 \right] + \frac{4N \rho^2}{g} - \frac{Nm^2 \rho}{g} + \frac{Nm^4}{16g}$$

Original fields now quadratic.

Coleman et al. (1974)
Back to the Future?

• If we integrate over the x’s in the most naive and unjustified way, we obtain the large-N effective potential for $\rho$.
• It is essentially identical to our previous result.

$$V_{eff}/N = \frac{4\rho^2}{g} - \frac{m^2\rho}{g} + \sqrt{2}\rho + \frac{m^4}{16g}$$

$$\rho = g\sigma^2$$

$$V_{eff}/N = -m^2\sigma^2 + 4g\sigma^4 + \frac{1}{2}\sqrt{32g\sigma^2} - \sqrt{2g}\sigma$$
PT-Symmetric Field Theories

- If we boldly extend this reasoning to field theory....

\[ V_{\text{eff}}/N = \frac{4\rho^2}{g} - \frac{m^2\rho}{g} + \frac{m^4}{16g} + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln [k^2 + 8\rho] \]

- Of course, this gives asymptotic freedom in the large-N limit..

\[ \beta = -\frac{g^2}{2\pi^2} \]
Conclusions for O(N) Models

• We now know the Hermitian form of O(N)-invariant PT-symmetric quantum mechanics. Its form has many unusual features.
• Its large-N limit can be derived in a simple way that we cannot yet justify.
• Arguments suggest that PT-symmetric field theories are asymptotically free, at least in the large-N limit.