Non-Hermitian quantum mechanics of strongly correlated systems

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References
Non-Hermitian quantum mechanics

Non-Hermitian generalization

**Tight-binding approximation**

\[
H_k = -t \sum_i \left( e^g c_{i+1}^\dagger c_i + e^{-g} c_i^\dagger c_{i+1} \right)
\]

**Continuum**

\[
H_k = \frac{\left( \hat{p} + ig(x) \right)^2}{2m}
\]

\[ g(x) : \text{Imaginary vector potential} \]
Why we consider the non-Hermitian generalization?

To calculate the length scale of the Hermitian systems from non-Hermitian energy spectra.

Non-Hermitian 1D random Anderson model

\[ H = -t \sum_{i=1}^{L} (e^{g} |i+1\rangle\langle i| + e^{-g} |i\rangle\langle i+1|) + \sum_{i=1}^{L} V_{i} |i\rangle\langle i| \]

N. Hatano and D. R. Nelson, PRL 77, 570 (1996); PRB 56, 8651 (1997)

A non-Hermitian critical point $g_c$ where the eigenvalue becomes complex.

Inverse localization length
Non-Hermitian generalization of strongly correlated quantum systems.

[Non-Hermitian critical point where the energy gap vanishes] = [Inverse correlation length]

- S=1/2 FM isotropic XY chain in a magnetic field
- S=1/2 AFM XXZ chain
- Majumdar-Ghosh model
- Hubbard model in the half-filled case
Outline of my talk

2. Why we can obtain the correlation length from non-Hermitian energy spectra?

[Hermitian]

\[ H = \sum_{-\pi < k < \pi} \varepsilon(k) \eta_k^\dagger \eta_k \]

[Non-Hermitian]

\[ H = \sum_{-\pi < k < \pi} \varepsilon(k + ig) \eta_k^\dagger \eta_k \]

\[ g_c = 1 / \xi \]

A zero of the dispersion relation

Outline of my talk

Why we can obtain the correlation length from non-Hermitian energy spectra?

[Hermitian]

\[ H = \sum_{-\pi < k < \pi} \varepsilon(k) \eta_k^\dagger \eta_k \]

[Non-Hermitian]

\[ H = \sum_{-\pi < k < \pi} \varepsilon(k + ig) \eta_k^\dagger \eta_k \]

[Hermitian] \[ g_c = 1/\xi \]

A zero of the dispersion relation

Applications to unsolved models

Heisenberg chain with nearest- and next-nearest-neighbor interactions

$$H = J \sum_{i=1}^{L} \left[ \vec{S}_i \cdot \vec{S}_{i+1} + \alpha \vec{S}_i \cdot \vec{S}_{i+2} \right]$$

[Extrapolated estimates of $g_c$] = [Inverse correlation length]
Non-Hermitian Hubbard model

1D non-Hermitian Hubbard model (half filled)

\[ H = - \sum_i \sum_{\sigma=\uparrow,\downarrow} (te^g c_{i+1,\sigma}^\dagger c_{i,\sigma} + te^{-g} c_{i,\sigma}^\dagger c_{i+1,\sigma}) + \sum_i U c_{i,\uparrow}^\dagger c_{i,\uparrow} c_{i,\downarrow}^\dagger c_{i,\downarrow} \]

T. Fukui and N. Kawakami, PRB 58, 16051 (1998)

[The g dependence of the Hubbard gap for U/t=2]
Non-Hermitian Hubbard model

The non-Hermitian critical point $g_c$ where the Hubbard gap vanishes

$$g_c = \arcsinh(U/4t) + 2i \int_{-\infty}^{\infty} \arctan \left( \frac{\lambda + iU/4t}{U/4t} \right) \sigma(\lambda) d\lambda$$

$$\sigma(\lambda) = \frac{1}{2\pi} \int_{0}^{\infty} \text{sech}((U/4t)\omega) \cos(\lambda\omega) J_0(\omega) d\omega$$


Inverse correlation length of the charge excitation

$$\frac{1}{\xi} = \arcsinh(U/4t) - 2 \int_{0}^{\infty} \frac{\sinh((U/4t)\omega) J_0(\omega)}{\omega(1 + e^{2(U/4t)\omega})} d\omega$$

Physical meaning of the non-Hermitian Hubbard model

1. \( \text{Im} \, k = 1 / \xi_{\text{charge}} \) with \( \varepsilon(k) = 0 \) [for Hermitian systems]

2. By our non-Hermitian generalization,

\[
H = \sum_{-\pi < k < \pi} \varepsilon(k) \eta_k^\dagger \eta_k \quad \text{and} \quad H = \sum_{-\pi < k < \pi} \varepsilon(k + ig) \eta_k^\dagger \eta_k
\]
A zero of the dispersion relation

[Zeros of the dispersion relation in the momentum space]

\[ H = -t \sum_i \sum_{\sigma=\uparrow,\downarrow} (c_i^{\dagger,\sigma} c_{i+1,\sigma} + c_i^{\dagger,\sigma} c_{i+1,\sigma}^\dagger) + \sum_i U n_i,\uparrow n_i,\downarrow \]

Excitation energy
\[ E(k_h) = U + 4t \cos k_h + 8t \int_0^\infty \frac{\cos(\omega \sin k_h) J_1(\omega)}{\omega(1 + e^{\omega U/2t})} d\omega \]

Momentum
\[ p(k_h) = k_h + 2 \int_0^\infty \frac{\sin(\omega \sin k_h) J_0(\omega)}{\omega(1 + e^{\omega U/2t})} d\omega \]
A zero of the dispersion relation

Zeros of the dispersion relation in the complex *quasimomentum* space.

\[ k_h = \pm \pi + i \arcsinh\left(\frac{U}{4t}\right) \quad E(k_h) = 0 \]

Zeros of the dispersion relation in the complex *momentum* space.

\[ p(k_h) = \pm \pi + i \left[ \arcsinh\left(\frac{U}{4t}\right) - 2 \int_0^\infty \frac{\sin(\omega U/4t) J_0(\omega)}{\omega(1 + e^{\omega U/2t})} d\omega \right] \]

\[ \frac{1}{\xi} \]

Physical meaning of the non-Hermitian Hubbard model

The non-Hermiticity $g$ shifts the momentum $k$ to $k + ig$.

Bethe-ansatz equation for infinite systems

$$g = 2\pi i z_c(k) - i \left[ k + 2\int_0^\infty \frac{\sin(\omega \sin k) J_0(\omega)}{\omega(1 + \exp(\omega U/2t))} d\omega \right]$$

$\Re p(k) = 2\pi z_c(k)$

$\Im p(k) = g$

$z_c(k)$: quantum # of the quasimomentum $k$

$z_c(k) \equiv I_j/L$

Ground state

$[I_j = (L-1)/2,(L-3)/2,\ldots,-(L-1)/2]$
Physical meaning of the non-Hermitian Hubbard model

The distribution of the momentum of the hole with $U/t=4$.

Zeros of the dispersion relation

$g=0.245$
$g=0.2$
$g=0.15$
$g=0.1$
$g=0.05$
$g=0.0$
Non-Hermitian AFM XXZ chain

\[ H = J \sum_{i=1}^{L} \left[ \frac{1}{2} (e^{2g} S_i^- S_{i+1}^+ + e^{-2g} S_i^+ S_{i+1}^-) + \Delta S_i^z S_{i+1}^z \right] \quad (J > 0, \Delta > 1) \]


\[ H = J \sum_{i=1}^{L} \left[ \frac{1}{2} (e^{2g} S_i^- S_{i+1}^+ + e^{-2g} S_i^+ S_{i+1}^-) + S_i^z S_{i+1}^z \right] \quad \left( J \equiv \frac{4t^2}{U} \right) \]

Non-Hermitian Hubbard model in order to vanish the spin gap

\[ H = -\sum_{i} \sum_{\sigma=\uparrow,\downarrow} (te^{g\sigma} c_{i+1,\sigma}^{\dagger} c_{i,\sigma} + te^{-g\sigma} c_{i,\sigma}^{\dagger} c_{i+1,\sigma}) + \sum_{i} Uc_{i,\uparrow}^{\dagger} c_{i,\uparrow} c_{i,\downarrow}^{\dagger} c_{i,\downarrow} \]
Non-Hermitian AFM XXZ chain

The non-Hermitian critical point $g_c$ where the energy gap vanishes

$$g_c = \frac{\gamma}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{\tanh(n\gamma)}{n}, \quad \text{where} \quad \gamma = \text{arccosh} \Delta$$

The inverse correlation length of the spinon excitation

R. J. Baxter, Exactly Solved Models in Statistical Mechanics

Physical meaning of the non-Hermitian generalization

$$p(\lambda_1, \lambda_2) \rightarrow p(\lambda_1, \lambda_2) + 2ig$$
Zeros of the dispersion relation of AFM XXZ chain

\[ H = J \sum_{i=1}^{L} \left[ \frac{1}{2} (S_i^- S_{i+1}^+ + S_i^+ S_{i+1}^-) + \Delta S_i^z S_{i+1}^z \right] \]

\[ (J > 0, \Delta > 1) \]

Zeros of the dispersion relation of the spinon excitation at the spin rapidities \( \lambda = \lambda_1, \lambda_2 \)

quasi-momentum \( \lambda_1 = \lambda_2 = \pm \pi / \gamma + i \)

momentum \( p(\lambda_1, \lambda_2) = \pm \pi + i[\gamma + 2 \sum_{n=1}^{\infty} (-1)^n \tanh(n\gamma) / n] \)

\[ = \pm \pi + i[2 / \xi] = \pm \pi + i[2g_c] \]

Frustrated quantum spin chain

Applications to systems whose dispersion relations are not exactly obtained.

\[ S=1/2 \text{ AFM Heisenberg chain with NNN interactions} \]

\[ H = J \sum_{i=1}^{L} \left[ \vec{S}_i \cdot \vec{S}_{i+1} + \alpha \vec{S}_i \cdot \vec{S}_{i+2} \right] \]

Ground-state properties (0 ≤ \( \alpha \) ≤ 1/2)

\( \alpha = 0 \)  \( \Rightarrow \) Spin liquid  \( \xi: \) infinite

\( \alpha = \alpha_c \approx 0.2411 \)  \( \Rightarrow \) Spin dimer (two-fold degenerate)  \( \Rightarrow \xi: \) finite

\( \alpha = 1/2 \)  \( \Rightarrow \) Majumdar-Ghosh model  \( \xi = 2 / \ln 2 \)
Non-Hermitian analysis of NNN Heisenberg chains

Non-Hermitian NNN Heisenberg chain

\[ H = J \sum_{i=1}^{L} \left[ \frac{1}{2} (e^{2g} S_i^+ S_{i+1}^- + e^{-2g} S_i^- S_{i+1}^+) + S_i^z S_{i+1}^z \right] + \alpha J \sum_{i=1}^{L} \left[ \frac{1}{2} (e^{4g} S_i^- S_{i+2}^+ + e^{-4g} S_i^+ S_{i+2}^-) + S_i^z S_{i+2}^z \right] \]

The a dependence of the spectrum flows per site for 8 sites

Finite-size plot of \( g_c(L) \)
\(\alpha\) dependence of \(g_c(L)\)

\[1/\xi = \ln 2 / 2 \approx 0.3466\]

(Finite-size scaling of the correlation func.)

Consistent!

\[g_c \sim 0.3472\text{(linear)}, \ g_c \sim 0.3463\text{(2nd order)}\]

Massive-massless transition point around \(\alpha \sim 0.25\) (\(\alpha_c \sim 0.2411\))
Summary

1. The non-Hermitian generalization of SCQS
   \[ g_c = \frac{1}{\xi} \]

2. Physical meaning of the non-Hermitian generalization
   Non-Hermitian generalization: \[ \varepsilon(k) \rightarrow \varepsilon(k + ig) \]

3. \[ \text{Im } k = \frac{1}{\xi} \text{ with } \varepsilon(k) = 0 \]

[Conjecture]
Applicable to general quantum systems?