Non-Hermitian von Roos Hamiltonian’s
η-weak-pseudo-Hermiticity and exact solvability

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Abstract

A complexified von Roos Hamiltonian is considered and a Hermitian
first-order intertwining differential operator is used to obtain the related
position dependent mass η-weak-pseudo-Hermitian Hamiltonians. Two
"user-friendly" reference-target maps are introduced to serve for exact-
solvability of some non-Hermitian η-weak-pseudo-Hermitian position de-
pendent mass Hamiltonians. A non-Hermitian PT-symmetric Scarf II
and a non-Hermitian periodic-type PT-symmetric Samsonov-Roy poten-
tials are used as reference models in a "user-friendly" reference-target
map and the corresponding isospectral Hamiltonians are obtained. It is
observed that for each exactly-solvable reference Hamiltonian there is a
corresponding set of exactly-solvable target Hamiltonians.

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1 Introduction

Subjected to von Roos constraint α + β + γ = −1; α, β, γ ∈ ℜ, the von Roos
position-dependent-mass (PDM) Hamiltonian [1-12] reads

\[ H = -\partial_x \left( \frac{1}{M(x)} \right) \partial_x \tilde{V}(x), \] (1)

with

\[ \tilde{V}(x) = \frac{1}{2} (1 + \beta) \frac{M''(x)}{M(x)} - [\alpha (\alpha + \beta + 1) + \beta + 1] \frac{M'(x)^2}{M(x)^2} + V(x), \] (2)

and primes denote derivatives. An obvious profile change of the potential \( \tilde{V}(x) \)
obtains as \( \alpha, \beta, \) and \( \gamma \) change, manifesting in effect an ordering ambiguity con-
flict in the process of choosing a unique kinetic energy operator

\[ T = -\frac{1}{2} \left[ M(x)^\alpha \partial_x M(x)^\beta \partial_x M(x)^\gamma + M(x)^\gamma \partial_x M(x)^\beta \partial_x M(x)^\alpha \right] \]  

(3)

Hence, \( \alpha, \beta, \) and \( \gamma \) are usually called the von Roos ambiguity parameters. Yet, such PDM-quantum-particles (i.e., \( M(x) = m_x m(x) \)) are used in the energy density many-body problem, in the determination of the electronic properties of semiconductors and quantum dots [1-5].

Regardless of the continuity requirements on the wave function at the boundaries of abrupt heterojunctions between two crystals [6] and/or Dutra’s and Almeida’s [7] reliability test, there exist several suggestions for the kinetic energy operator in (3). We may recollect the Gora’s and Williams’ (\( \beta = \gamma = 0, \alpha = -1 \)) [8], Ben Danial’s and Duke’s (\( \alpha = \gamma = 0, \beta = -1 \)) [9], Zhu’s and Kroemer’s (\( \alpha = \gamma = -1/2, \beta = 0 \)) [10], Li’s and Kuhn’s (\( \beta = \gamma = -1/2, \alpha = 0 \)) [11], and the very recent Mustafa’s and Mazharimousavi’s (\( \alpha = \gamma = -1/4, \beta = -1/2 \)) [3]. Nevertheless, in this work we shall deal with these orderings irrespective to their classifications of being "good-" (i.e., satisfying the continuity requirements on the wave function, mentioned above, and surviving the Dutra’s and Almeida’s [7] reliability test) or "to-be-discarded-" orderings (i.e., not satisfying the continuity requirements on the wave function and/or failing the Dutra’s and Almeida’s [7] reliability test). The reader is advised to refer to, e.g., Mustafa and Mazharimousavi [3] for more details.

The growing interest in the non-Hermitian pseudo-Hermitian Hamiltonians with real spectra [13-21], on the other hand, have inspired our resent work on PDM first-order-intertwining operator and \( \eta \)-weak-pseudo-Hermiticity generators [12]. A Hamiltonian \( H \) is pseudo-Hermitian if it obeys the similarity transformation \( \eta H \eta^{-1} = H^\dagger \), where \( \eta \) is a Hermitian invertible linear operator and \( (\cdot)^\dagger \) denotes the adjoint. The existence of real eigenvalues is realized to be associated with a non-Hermitian Hamiltonian provided that it is an \( \eta \)-pseudo-Hermitian:

\[ \eta H = H^\dagger \eta, \]  

(4)

with respect to the nontrivial "metric"operator \( \eta = O^\dagger O \), for some linear invertible operator \( O : \mathcal{H} \to \mathcal{H} \) (\( \mathcal{H} \) is the Hilbert space). However, under some rather mild assumptions, we may even relax \( H \) to be an \( \eta \)-weak-pseudo-Hermitian by not restricting \( \eta \) to be Hermitian (cf., e.g., Bagchi and Quesne [17]), and linear and/or invertible (cf., e.g., Solombrino [18], Fityo [19], and Mustafa and Mazharimousavi [12,20]).

Whilst in the non-Hermitian pseudo-Hermitian Hamiltonians neighborhood [13-22], the non-Hermitian \( \mathcal{PT} \)-symmetric Hamiltonians (i.e., a Bender’s and Boettcher’s [13] initiative on the so called nowadays \( \mathcal{PT} \)-symmetric quantum mechanics) are unavoidably in point. They form a subclass of the non-Hermitian pseudo-Hermitian Hamiltonians (where \( \mathcal{P} \) denotes parity and \( \mathcal{T} \) mimics the time reversal). Namely, if \( \mathcal{PT} \mathcal{H} \mathcal{PT} = \mathcal{H} \) and if \( \mathcal{PT} \Phi(x) = \pm \Phi(x) \) the eigenvalues turn out to be real. However, if the latter condition is not satisfied the eigenvalues appear in complex-conjugate pairs (cf., e.g., Ahmed in [13]).
In this work, we consider (in section 2) a complexified von Roos Hamiltonian (1) (i.e., \( \tilde{V}(x) \rightarrow \tilde{V}(x) + iW(x) \)) regardless of the nature of the ordering of the ambiguity parameters as to being "good" or "to-be-discarded" ones. A Hermitian first-order differential PDM-intertwining operator is used to obtain the corresponding non-Hermitian \( \eta \)-weak-pseudo-Hermitian PDM-Hamiltonian. The related reference/old-target/new non-Hermitian \( \eta \)-weak-pseudo-Hermitian Hamiltonians’ map is also given in the same section. Yet, in connection with the resulting effective reference/old potential, two feasible "user-friendly" forms are suggested to serve for exact-solvability of some non-Hermitian \( \eta \)-weak-pseudo-Hermitian PDM-Hamiltonians. Such user-friendly forms turn out to imply that there is always a set of isospectral target/new non-Hermitian \( \eta \)-weak-pseudo-Hermitian PDM-Hamiltonians associated with "one" exactly-solvable reference/old non-Hermitian \( \eta \)-weak-pseudo-Hermitian PDM-Hamiltonian. In section 3, we use two illustrative examples (i.e., a complexified \( PT \)-symmetric Scarf-II and a periodic-type \( PT \)-symmetric Samsonov-Roy potentials) as reference/old models in one of the two "user-friendly" forms and report the corresponding sets of isospectral target/new non-Hermitian \( \eta \)-weak-pseudo-Hermitian PDM-Hamiltonians. Section 4 is devoted for the concluding remarks.

2 An \( \eta \)-intertwiner and \( \eta \)-weak-pseudo-Hermitian Hamiltonians’ reference-target map

A complexification of the potential \( \tilde{V}(x) \) in (1) may be achieved by the transformation \( \tilde{V}(x) \rightarrow \tilde{V}(x) + iW(x) \), where \( \tilde{V}(x), W(x) \in \mathbb{R} \) and \( \mathbb{R} \ni x \in (-\infty, \infty) \). Hence, Hamiltonian (1) becomes non-Hermitian and reads

\[
H = -\mu(x)^2 \frac{d^2}{dx^2} - 2\mu(x) \mu'(x) \frac{d}{dx} + \tilde{V}(x) + iW(x),
\]

with \( \mu(x) = \pm 1/\sqrt{M(x)} \). A Hermitian first-order intertwining PDM-differential operator (cf., e.g., Mustafa and Mazharimousavi [12] on the detailed origin of this PDM-operator) of the form

\[
\eta = -i \left[ \mu(x) \frac{d}{dx} + \mu'(x) / 2 \right] + F(x); \quad F(x), \mu(x) \in \mathbb{R}
\]

would result, when used in (4),

\[
W(x) = -\mu(x) F'(x),
\]

\[
\tilde{V}(x) = -F(x)^2 - \frac{1}{2} \mu(x) \mu''(x) - \frac{1}{4} \mu'(x)^2 + \alpha_o,
\]

where \( \alpha_o \in \mathbb{R} \) is an integration constant. One may then recast \( V(x) \) as

\[
V(x) = \alpha_o - F(x)^2 + \left( \frac{1}{2} + \beta \right) \mu(x) \mu''(x) + \left[ 4\alpha (\alpha + \beta + 1) + \beta + \frac{3}{4} \right] \mu'(x)^2.
\]
One should, nevertheless, be reminded that an anti-Hermitian first-order operator of the form \( \eta = \mu(x) \partial_x + \mu'(x)/2 + F(x) \) will exactly do the same job (cf., e.g., Mustafa and Mazharimousavi [12]). Moreover, as a result of this intertwining process, a non-Hermitian \( \eta \)-weak-pseudo-Hermitian Hamiltonian is obtained.

We may now consider our non-Hermitian \( \eta \)-weak-pseudo-Hermitian Hamiltonian in (5), along with (7) and (8), in the one-dimensional Schrödinger equation

\[
H \psi(x) = E \psi(x)
\]

and construct the so-called reference/old-target/new non-Hermitian \( \eta \)-weak-pseudo-Hermitian Hamiltonians’ map (equation (10) is the so-called target/new Schrödinger equation). A task that would be achieved by the substitution

\[
\psi(x) = \varphi(q(x))/\sqrt{\mu(x)},
\]

to imply, with the requirement

\[
q'(x) = 1/\mu(x)
\]

that removes the first-order derivative \( \partial_q \varphi(q) \), a so-called reference/old Schrödinger equation

\[
-\partial_q^2 \varphi(q(x)) + \left[ \tilde{V}_{\text{eff}}(q(x)) - E \right] \varphi(q(x)) = 0,
\]

where

\[
\tilde{V}_{\text{eff}}(q(x)) = (\beta + 1) \mu(x) \mu''(x) + [4\alpha(\alpha + \beta + 1) + \beta + 1] \mu'(x)^2 - F(x)^2 + \alpha - i\mu(x) F'(x).
\]

This effective reference/old potential suggests two "user-friendly" forms. The first of which can be achieved through the choice

\[
(\beta + 1) \mu(x) \mu''(x) + [4\alpha(\alpha + \beta + 1) + \beta + 1] \mu'(x)^2 = 0,
\]

to imply

\[
\tilde{V}_{\text{eff},1}(q) = \alpha - F(q)^2 - iF'(q).
\]

where

\[
\frac{dF(x)}{dx} = \frac{dF(q(x))}{dq} = \frac{dq(x)}{dx} \frac{dF(q)}{dq} = \frac{1}{\mu(x)} \frac{dF(q)}{dq},
\]

is used. Hence \( \mu'(x) \mu(x)^{\delta} = \text{const.} \) and

\[
\mu(x) = [C_1 x + C_2]^{1/(\delta + 1)}; \quad \delta = \left[ 4\alpha + 1 + \frac{4\alpha^2}{\beta + 1} \right],
\]

where \( C_1 \) and \( C_2 \) are two constants and \( C_1, C_2 \in \mathbb{R} \). Nevertheless, one should notice that the Ben Danial’s and Duke’s (\( \alpha = \gamma = 0, \beta = -1 \)) ordering (although \( \beta = -1 \) is not allowed by (17) but satisfies (15)) has already been discussed by
Mustafa and Mazharimousavi [12]. Hence, the Ben Danial’s and Duke’s ordering shall not be considered in the forthcoming studies. Moreover, under such mass settings, we may report that; for Gora’s and Williams’ (\(\beta = \gamma = 0, \alpha = -1\)) and Li’s and Kuhn’s (\(\beta = \gamma = -1/2, \alpha = 0\)) orderings \(\delta_{GW} = \delta_{LK} = 1\), for Zhu’s and Kroemer’s (\(\alpha = \gamma = -1/2, \beta = 0\)) ordering \(\delta_{ZK} = 0\), and for Mustafa’s and Mazharimousavi’s (\(\alpha = \gamma = -1/4, \beta = -1/2\)) ordering \(\delta_{MM} = 1/2\).

The second choice
\[
F(x) = \mu'(x) \implies \mu(x) = \int^x F(y) \, dy, \tag{18}
\]
on the other hand, would lead to
\[
\tilde{V}_{eff,2}(q) = -iF'(q) + (\beta + 1) F'(q) + [4\alpha (\alpha + \beta + 1) + \beta] F(q)^2 + \alpha_o, \tag{19}
\]
Obviously, a \(\beta = -1\) (consequently, \(\alpha = \gamma = 0\) by the von Roos constraint \(\alpha + \beta + \gamma = -1\)) would lead to (16) (Ben Danial’s and Duke’s ordering is to be discarded in the current study for the reasons mentioned above).

### 3 Isospectral PDMs with \(\mu'(x) \mu(x)^\delta = \text{const.}\)

It is evident that the position-dependent-mass \(M(x)\) under the current settings is strictly determined through (15) and consequently through (17) to read
\[
M(x) = \mu(x)^{-2} = [C_1 x + C_2]^{-2/\delta+1}. \tag{20}
\]
This form identifies a class of isospectral position-dependent-mass functions satisfying the effective reference/old potential \(\tilde{V}_{eff,1}(q)\) of (16), regardless of the form of the \(\eta\)-weak-pseudo-Hermiticity generator \(F(q)\), and implies
\[
q(x) = \int^x \mu(y)^{-1} \, dy = \begin{cases} 
\frac{(\delta+1)}{\delta} [C_1 x + C_2]^{\delta/(\delta+1)} & ; \text{for } \delta \neq 0 \\
\frac{1}{\alpha^2} \ln (C_1 x + C_2) & ; \text{for } \delta = 0
\end{cases}. \tag{21}
\]
Unlike the case we have very recently considered in [12], where Ben Danial’s and Duke’s ordering (i.e., \(\alpha = \gamma = 0, \beta = -1\)) was used and the position-dependent-mass was left arbitrary instead (but, of course, a positive-valued function).

Nevertheless, one should notice that the form of our \(\tilde{V}_{eff,1}(q)\) in (16) depends only on the choice of our \(\eta\)-weak-pseudo-Hermiticity generator \(F(q)\) (the choice of which should be oriented in such a way that an exactly-solvable \(\eta\)-weak-pseudo-Hermitian reference/old Hamiltonian is obtained). Therefore, a set of exactly-solvable target/new potentials of (14) would obtain and depends only on the class of the strictly determined position-dependent-mass functions in (20). Two illustrative examples are in order.
3.1 A complexified $PT$-symmetric Scarf-II model

Let us recollect (cf., e.g., Mustafa and Mazharimousavi [12]) that an $\eta$-weak-pseudo-Hermiticity generator of the form

$$ F(q) = -V_2 \text{sech} q \implies F'(q) = V_2 \text{sech} q \tanh q $$  

would yield (with $\alpha_\varepsilon = 0$) a reference/old effective complexified $PT$-symmetric Scarf-II potential of the form

$$ \tilde{V}_{\text{eff},1}(q) = -V_2^2 \text{sech}^2 q - iV_2 \text{sech} q \tanh q \; ; \; \Re \ni V_2 \neq 0. $$

(23)

Which, in turn, would imply a target/new effective potential of the form

$$ \tilde{V}_{\text{eff},1}(x) = -4V_2^2 \frac{f(x)^2}{(f(x)^2 + 1)^2} + 2iV_2 \frac{f(x)(f(x)^2 - 1)}{(f(x)^2 + 1)^2}, $$

(24)

where $f(x) = \pm \exp[q(x)]$, with $q(x)$ given in (21). In this case, the target/new effective potentials in (24) form a set of isospectral potentials the eigenvalues of which are readily reported in [12,17] as

$$ E_n = - \left[ |V_2| - n - \frac{1}{2} \right]^2 \; ; \; n = 0, 1, 2, \ldots, n_{\text{max}} < (|V_2| - 1/2). $$

(25)

3.2 A periodic-type $PT$-symmetric Samsonov-Roy model

We may also recycle our $\eta$-weak-pseudo-Hermiticity generator

$$ F(q) = -\frac{4}{3\cos^2 q - 4} - \frac{5}{4}, $$

(26)

that implies (with $\alpha_\varepsilon = 0$) an effective periodic-type $PT$-symmetric Samsonov’s and Roy’s [12,14] reference/old potential

$$ \tilde{V}_{\text{eff},1}(q) = -\frac{6}{[\cos q + 2i\sin q]^2} - \frac{25}{16} \; ; \; \Re \ni q \in (-\pi, \pi). $$

(27)

This results, in effect, a target/new effective potential of the form

$$ \tilde{V}_{\text{eff},1}(x) = -\frac{6}{[g(x) - 2i\mu(x)g'(x)]^2} - \frac{25}{16}, $$

(28)

where $g(x) = \cos(q(x))$, $\mu(x)$ and $q(x)$ are as given in (17) and (21), respectively. Hence, the set of target/new effective potentials in (28) are isospectral and the corresponding eigenvalues [12,14] are given by

$$ E_n = \frac{n^2}{4} - \frac{25}{16} \; ; \; n = 1, 3, 4, 5, \ldots, $$

(29)

with a missing $n = 2$ state (the details of which can be found in Samsonov and Roy [14]).
4 Concluding remarks

As long as \( \eta \)-weak-pseudo-Hermitian Hamiltonians are in point, their solvability-nature/type (i.e., e.g., exact-, quasi-exact-, conditionally-exact-, etc.) is still fresh and not yet adequately explored. Amongst is the \( \eta \)-weak-pseudo-Hermitian von Roos PDM-Hamiltonian. In this work, we tried to (at least) partially fill this gap and add a flavour into such solvability territories of the \( \eta \)-weak-pseudo-Hermitian Hamiltonians associated with position-dependent-mass settings.

We have suggested two "user-friendly" forms for the reference/old \( \eta \)-weak-pseudo-Hermitian PDM-Hamiltonians' map. Only one of which (i.e., \( \bar{V}_{eff,1}(q) \) of (16)) is exemplified through a non-Hermitian \( PT \)-symmetric Scarf II and a non-Hermitian \( PT \)-symmetric Samsonov-Roy periodic-type models. It is observed that for each of these models there is a set of exactly-solvable isospectral target/new \( \eta \)-weak-pseudo-Hermitian PDM-Hamiltonians (documented in (24) for Scarf II and in (28) for Samsonov-Roy). However, we were unlucky to find any illustrative example that can be classified as "successful" for the "user-friendly" form \( \bar{V}_{eff,2}(q) \) in (19). Nonetheless, the corresponding target/new isospectral set of \( \eta \)-weak-pseudo-Hermitian PDM-Hamiltonians is anticipated to be feasibly large (as documented by (18)) and not restricted to the position-dependent-mass form (unlike the case of \( \bar{V}_{eff,1}(q) \) in (16), which is restricted to the position-dependent-mass function \( M(x) \) in (20)).

Moreover, we may report that a generating function \( F(q) = a \exp(-q) \) would lead to (with \( \alpha_0 = 0 \)) to

\[
\bar{V}_{eff,1}(q) = -a^2 \exp(-2q) + ia \exp(-q) \tag{30}
\]

of (16), and

\[
\bar{V}_{eff,2}(q) = a^2 [4\alpha (\alpha + \beta + 1) + \beta] \exp(-2q) - a (\beta + 1 - i) \exp(-q) \tag{31}
\]

of (19). The bound-states of the former (30) (a non-Hermitian Morse model) are reported to form an empty set of eigenvalues and, hence, labeled as "unfortunate" for it leads to an empty set of "unfortunate" isospectral \( \eta \)-weak-pseudo-Hermitian target PDM-Hamiltonians (cf., e.g., Mustafa and Mazharimousavi [12], Bagchi and Quesne [23], and Ahmed [24]). The latter (31), on the other hand, does not fit into any of the "so-far-known" exactly-solvable non-Hermitian Morse-type models, to the best of our knowledge.

Finally, one may add that the current strictly-determined set of target/new effective potentials \( \bar{V}_{eff,1}(x) \) in (24) forms a subset of the target/new effective potentials reported in equations (25) and (26) by Mustafa and Mazharimousavi [12]. Similar trend is also observed for \( \bar{V}_{eff,1}(x) \) in (28) as it forms a subset of the effective potentials in equations (34) and (35) of [12]. Hence, the scenario of the energy-levels crossing and the feasible manifestation of the flown away states discussed in [12] remains effective, as long as the our two illustrative examples are concerned.
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