Spectral Properties of an Eigenvalue Problem due to Richardson

Paul E. Shanley

London

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Richardson Problem (1918)

- Indefinite Sturm – Liouville (SL) problem.
- Behavior of the spectrum is :

  “amazing”, (Atkinson, Jabon(1984))

  “complicated and strange”, (Kong, Wu, Zettl (2001)).
- To atomic physicists it is a Sturmian problem.
- Therefore closely related to a Schrodinger problem.

- R, 1918 Amer. J. Math. **40** 283
- AJ, 1984, in H. G. Kaper and A. Zettl, Eds., *Spectral Theory of Sturm –Liouville Differential Operators*, ANL – 84 – 73, Argonne Nat. Lab., Argonne, Ill.
- KWZ, 2001 J. Diff. Eqs. **177** 1

Spectral Inversion
Schrodinger ⟷ Sturmian

- Schrodinger $V(x) = \lambda v(x)$

$$\lambda = Independent\ Variable$$

$$E_n(\lambda) = Dependent\ Variable$$ = *Eigenvalue*

- Sturmian = Richardson

$$E = Independent\ Variable$$

$$\lambda_n(E) = Dependent\ Variable$$ = *Eigencoupling*

Each function takes its values on a Riemann surface.

Richardson Problem – Right Indefinite SL Problem

$$- \phi_n''(E,x) - E\phi_n(E,x) = \lambda_n(E)\,\mathrm{sgn}(x)\,\phi_n(E,x)$$

$$-1 \le x \le +1, \qquad \phi_n(E,\pm 1) = 0\ .$$

$$\mathrm{sgn}(x) = -1, \quad x < 0,$$

$$\mathrm{sgn}(x) = +1, \quad x > 0,$$

$$\mathrm{sgn}(x) = \text{Indefinite weight}$$

Schrodinger Problem

$$\lambda_n(E) \to \lambda$$

$$E \to E_n(\lambda)$$

$$\phi_n(E,x) \to \psi_n(\lambda,x)$$

$$-\psi_n''(\lambda,x) - \lambda\,\mathrm{sgn}(x)\,\psi_n(\lambda,x) = E_n(\lambda)\psi_n(\lambda,x)$$

then $V(x) = -\lambda\,\mathrm{sgn}(x)$

↑ 1

$$\lim_{\lambda \to 0} E_n(\lambda) = (\pi^2/4)\,n^2, \quad n=1,2,3,\cdots$$

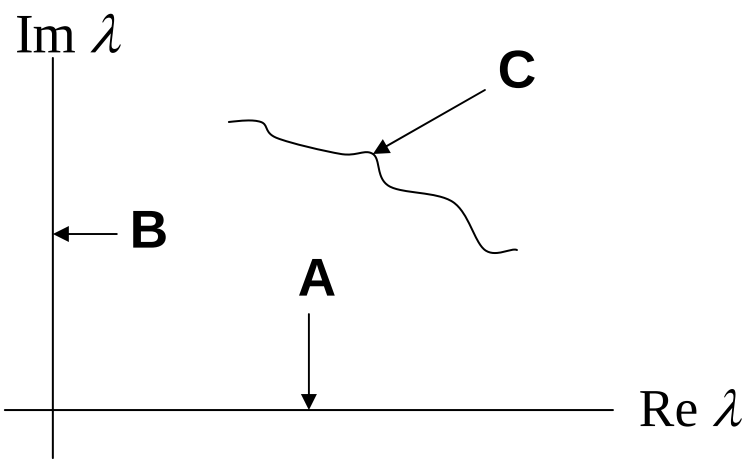
Znojil (2001) and Znojil, Revai (2001) studied the Schrodinger problem for Re $\lambda = 0$.

Z, 2001 Phys. Lett. **A285** 7

ZR, 2001 Mod. Phys. Lett. **16** 2273

In a Schrodinger problem, where in complex coupling are the eigenvalues real ?

$$\text{Im } E_n(\lambda) = 0$$



Real Version

$\text{Im } E_n \rightarrow F$

$\text{Re } \lambda \rightarrow x$

$\text{Im } \lambda \rightarrow y$

$F(x,y) = 0$

$y(x) = \text{curve}$

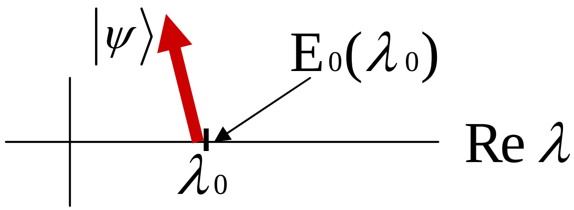
A, B, and C are each a real locus.

A) Real Axis

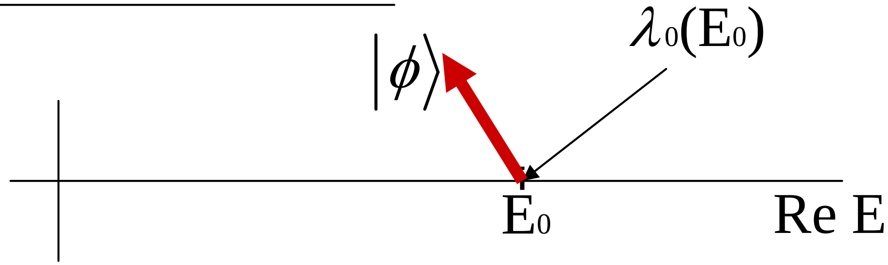
B) Imag Axis

C) Curve

Remarks on Spectral Inversion



Schrodinger



Richardson

- Under spectral inversion: Place $\rightarrow$ Value

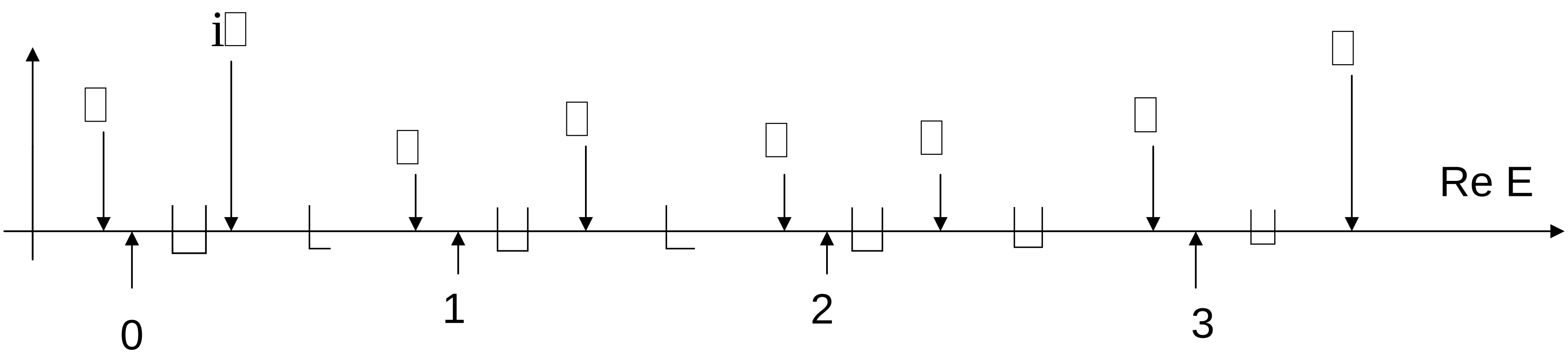
  Value $\rightarrow$ Place

- $E_0'(\lambda_0) = 0 \;\Rightarrow\; \lambda_0'(E_0) = \infty$

  $E_0''(\lambda_0) \neq 0$

  Schr. critical points $\Rightarrow$ Rich. square roots

Richardson Spectrum,(Atkinson and Jabon, 1984)



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Four Complexities

(1) Three Spectral Types: □ , i□ , □

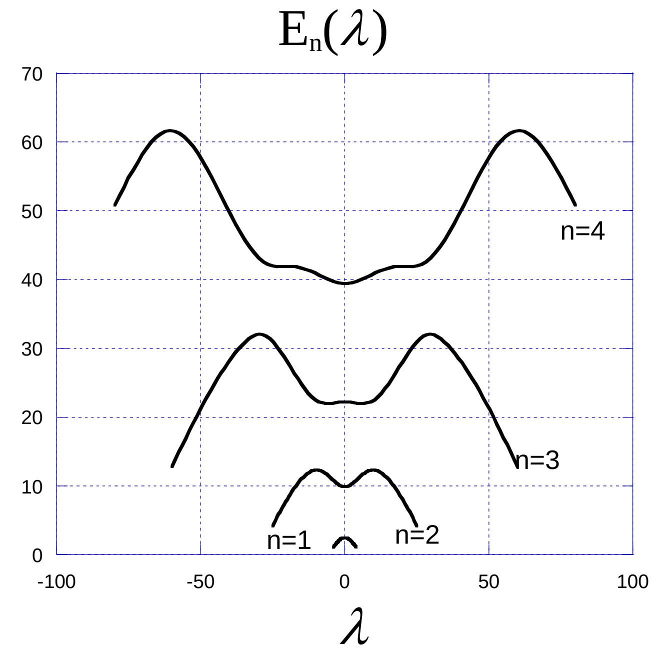
(2) Changing Oscillation Count : 0 to 1 to 2 to 3 to …..

(3) Each dot is a square root singularity which implies many sheets.

(4) The inverse of this must be a Schrodinger function.

Numerical Results

$$V(x) = -\lambda \, \mathrm{sgn}(x)$$



Binding, Volkmer 1996

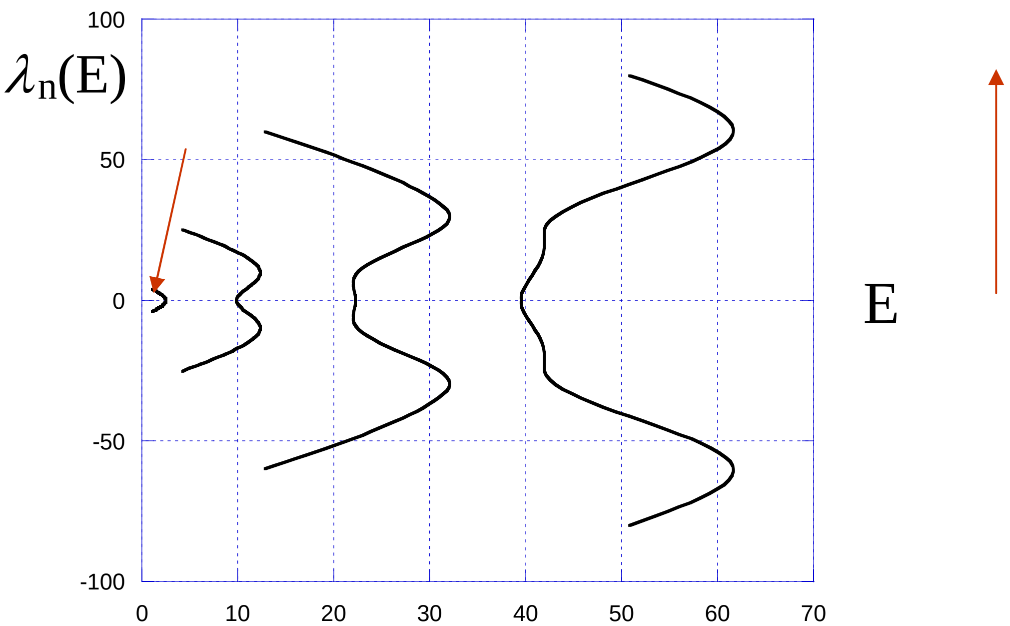
(1) $E_n(-\lambda) = E_n(\lambda)$

(2) $E_n(\lambda^*) = E_n^*(\lambda)$

(3) $E_n(\lambda)$ has $2n-1$ critical points

where $E_n'(\lambda) = 0$.

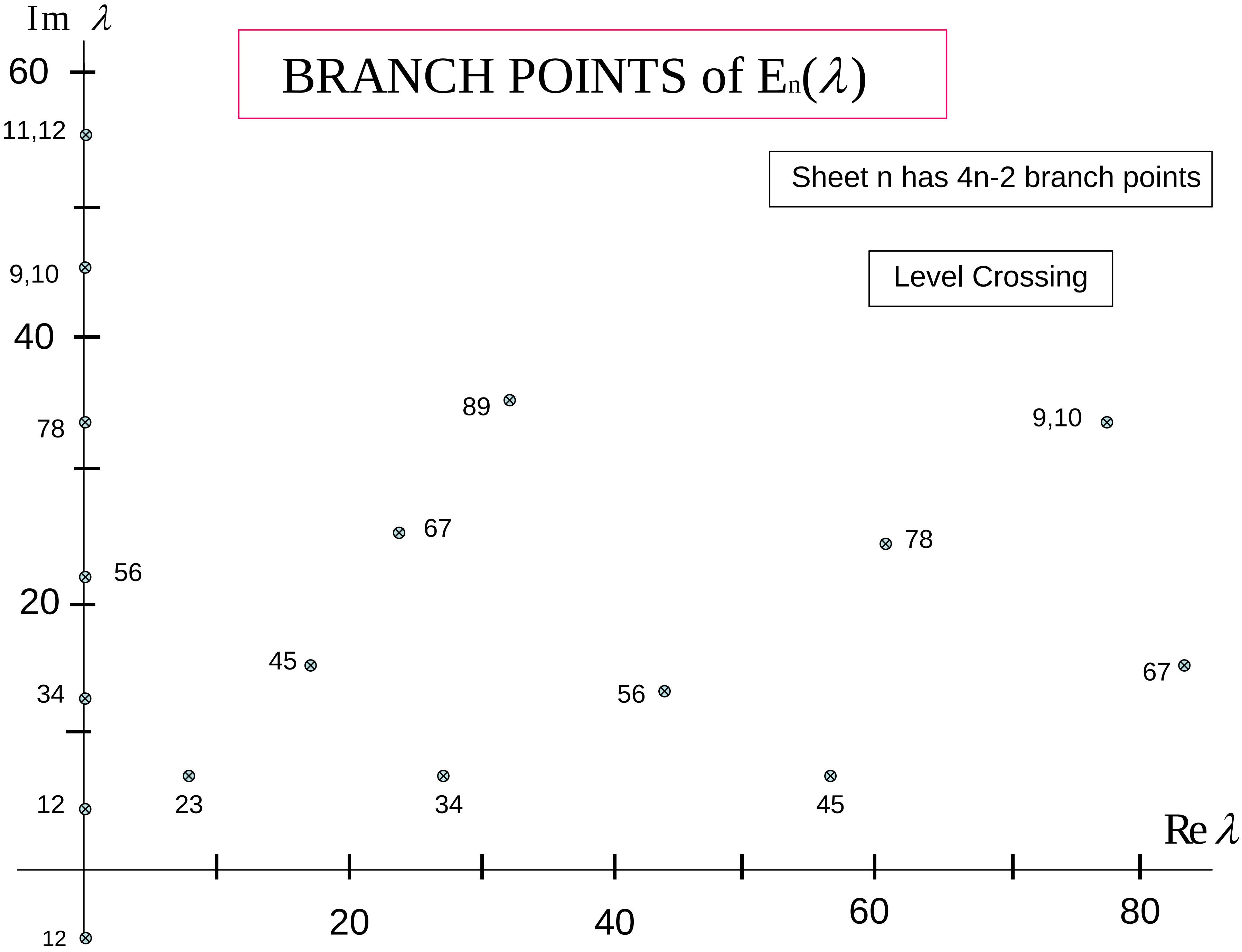
Richardson Graph



BRANCH POINTS of $E_n(\lambda)$

Sheet n has 4n-2 branch points

Level Crossing



Complexity No. 1

Schrodinger

On A, B, or C. → (S. I.) → Richardson

$$\underline{Values}: \ E_n(\lambda) \in \Box \quad \longrightarrow \quad \underline{Places}: E \in \Box$$

$$\underline{Places}: \lambda \in \Box, \ i\Box, \ \Box \quad \longrightarrow \quad \underline{Values}: \ \lambda_n(E) \in \Box, \ i\Box, \ \Box$$

Complexity No. 2

The changing oscillation count in Richardson

implies

level crossing in the Schrodinger problem.

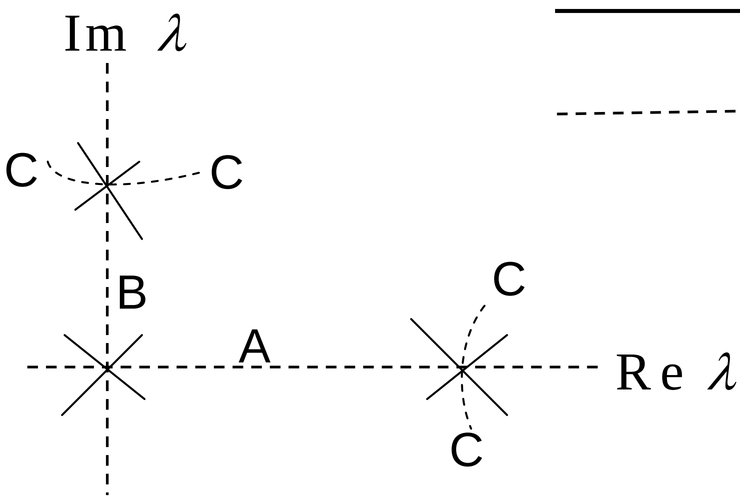
Can a C touch an A or a B ? Yes, at a critical point.

Assume:

(1) $E_n(-\lambda) = E_n(\lambda)$.

(2) Real locus A and B exist.

(3) Put a critical point on A and B.



———— $\mathrm{Re}\, E_n(\lambda) = \text{constant}$

- - - - - $\mathrm{Im}\, E_n(\lambda) = 0$

Conclusion: Critical points are "sources" of B and C.

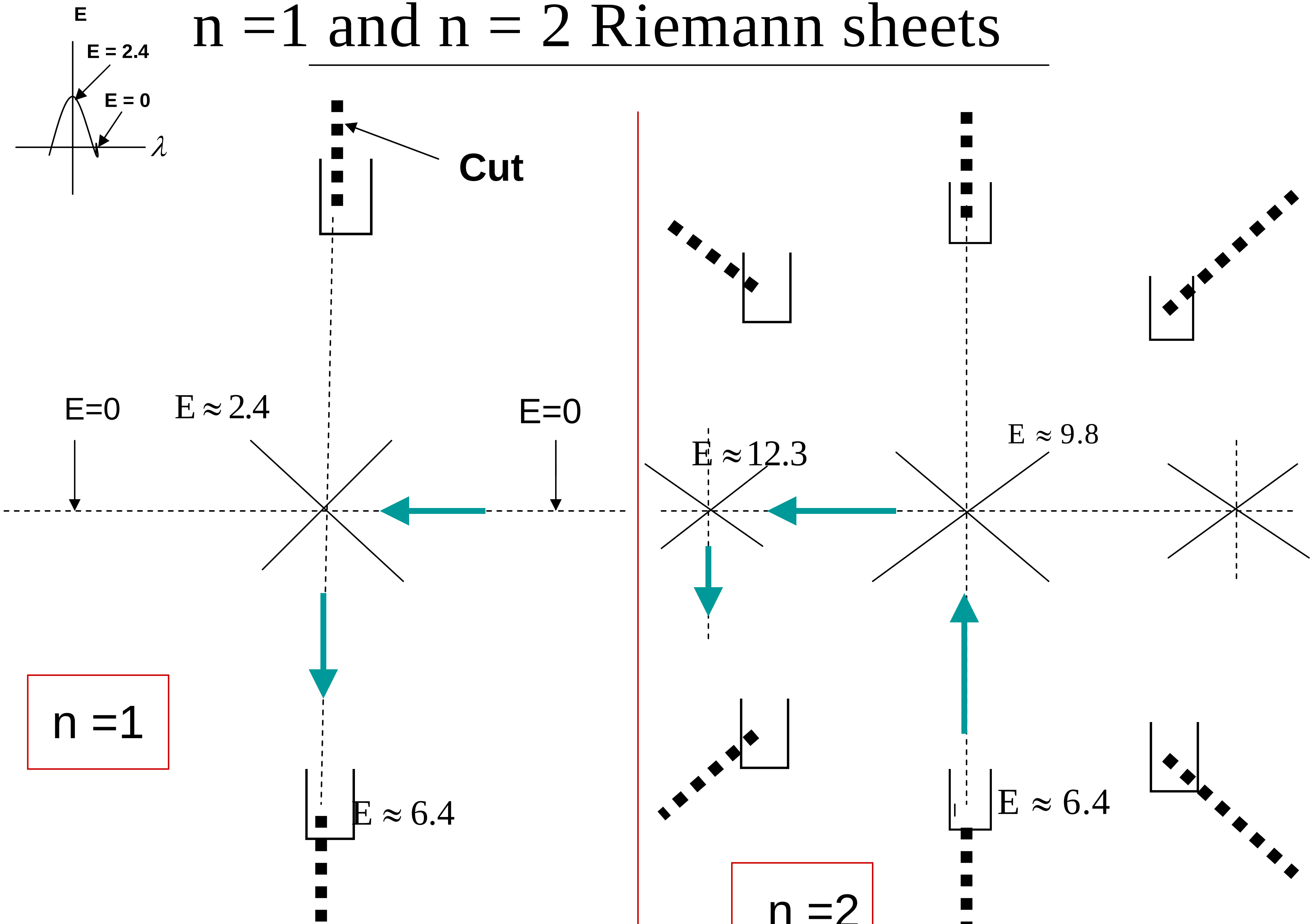
Can we match

Richardson values / places

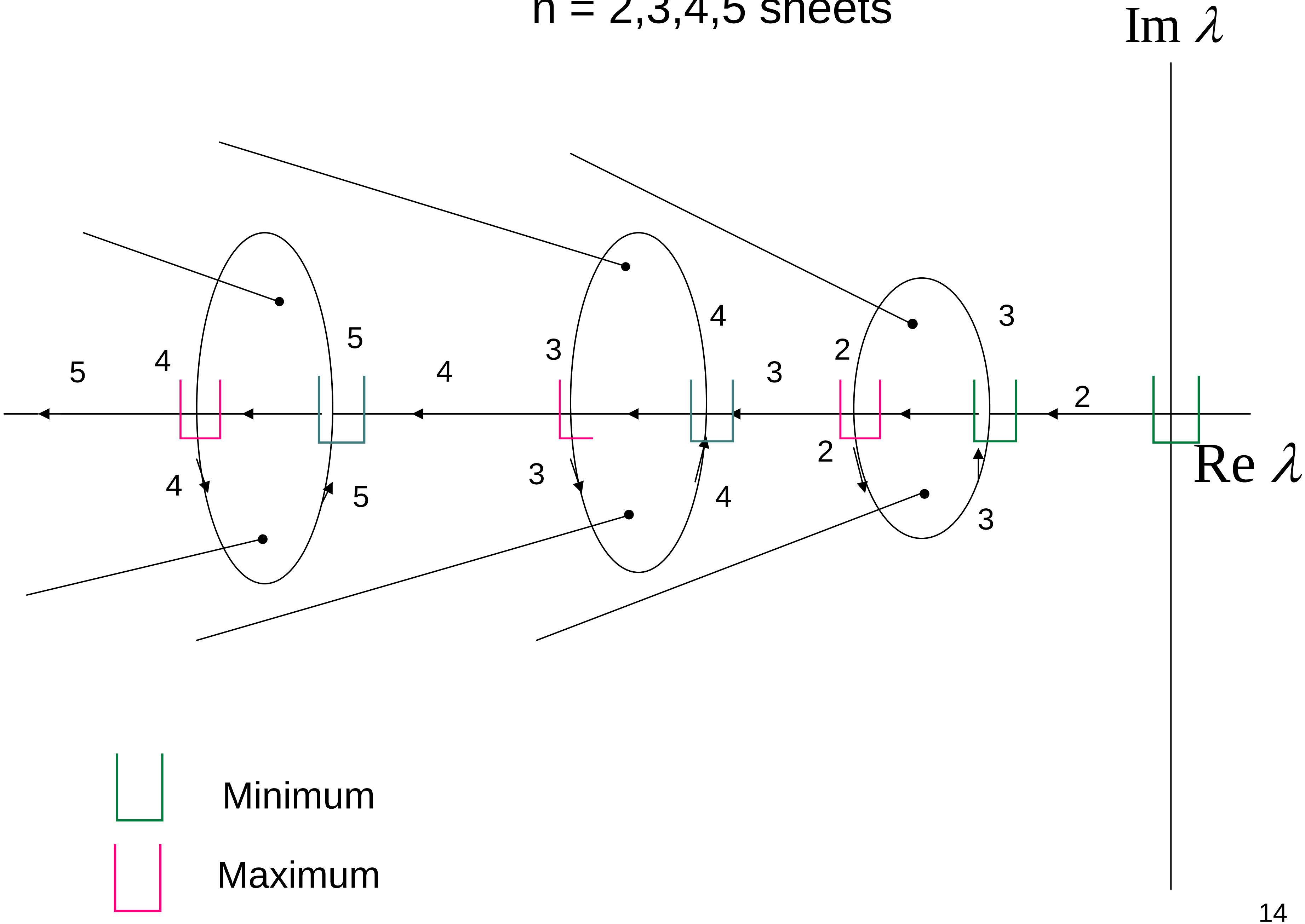
with

Schrodinger places / values ?

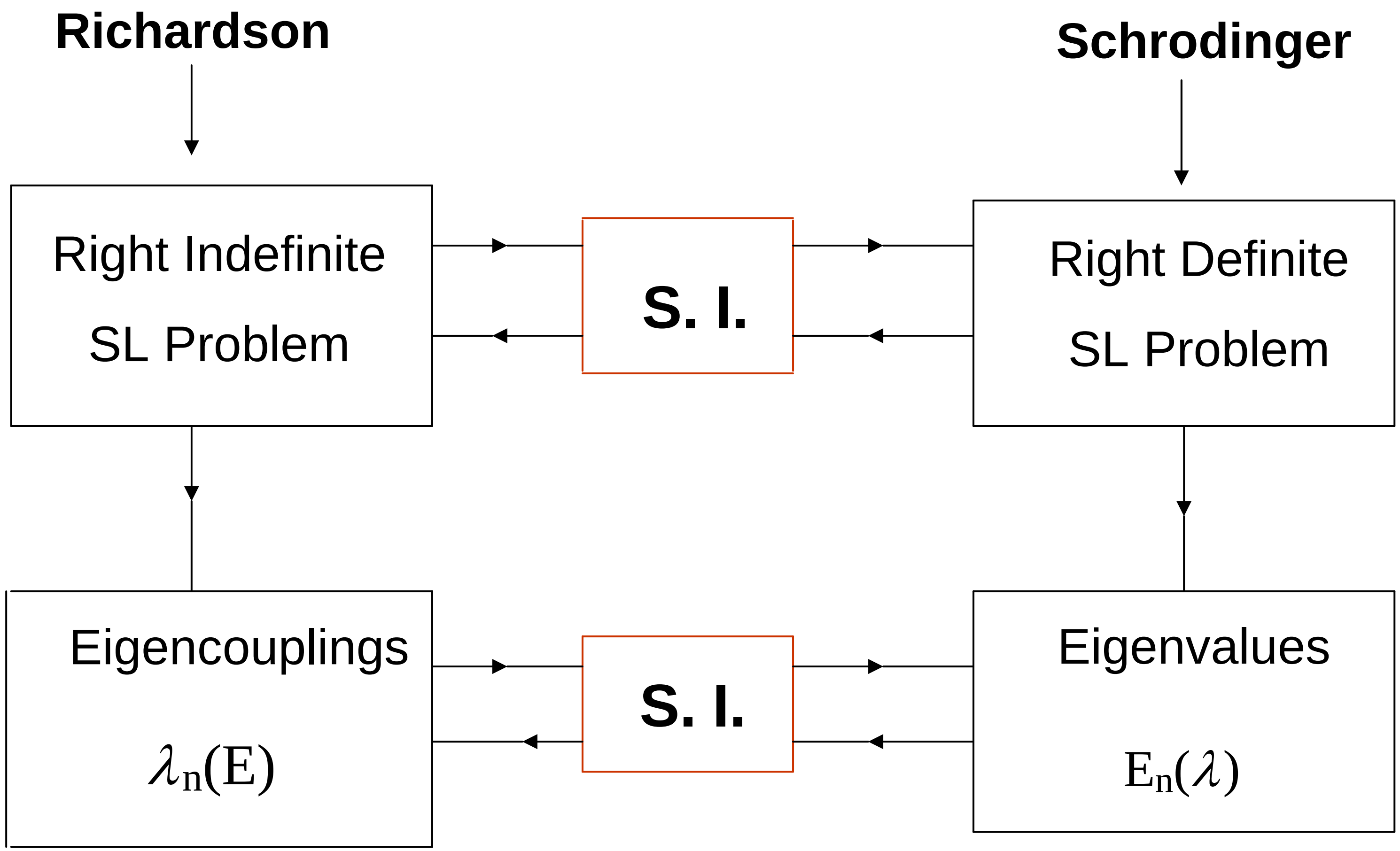
n =1 and n = 2 Riemann sheets



n = 2,3,4,5 sheets



Summary


Richardson
Schrodinger
Right Indefinite
SL Problem
S. I.
Right Definite
SL Problem
Eigencouplings
$\lambda_n(E)$
S. I.
Eigenvalues
$E_n(\lambda)$

Other Schrodinger Problems

Potential	B's ?	C's ?
$\lambda x^2 + x^4$	NO	NO
$\lambda x + x^4$	YES	YES
2 q cos (2x) ↓	**Mathieu's Equation**	
Even, π	YES	YES
Odd, π	YES	YES
Even, 2π	NO	YES
Odd, 2π	NO	YES

Herglotz (H) Anti-Herglotz (AH) Functions

Definition

$E_n(\lambda)$ is (H/AH) if

$$\text{Im } E_n(\lambda) = K_n(\lambda) \text{ Im } \lambda \ ,$$
$$\text{with } K_n(\lambda) \ (> 0/< 0\ ), \text{ all } n, \text{ all } \lambda \in \square \ .$$

If $E_n(\lambda)$ is H or AH, then

(1) B, C, and PT are impossible.

(2) $E'_n(\lambda) \neq 0$ for λ,E real ,

$$\Rightarrow \ E_n(\lambda) \text{ cannot be an even function.}$$

(3) If $E_n(\lambda)$ is H or AH, so is $\lambda_n(E)$.

(4) For $V(x) = \lambda\, x^2 + x^4$, $E_n(\lambda)$ is H,(Simon, 1970)

Herglotz Case

Take $\mathrm{Im} \int dx\ \psi_n^*(x) \otimes$ (Schr. Eq.)

with $\langle \psi | \psi \rangle = 1$

Then $$\boxed{\mathrm{Im}\ E_n(\lambda) = \mathrm{Im}\ \lambda \int dx \left|\psi_n(\lambda, x)\right|^2 v(x)}$$

If $v(x) \geq 0$, $K_n(\lambda) > 0$, H case.

If $v(x) \leq 0$, $K_n(\lambda) < 0$, AH case.

Alternatively in either case the Schrodinger and Sturmian problems are right-definite, they both have real eigenvalues with no B or C.

Real Locus Locator

If a problem is not H or AH it is still true that

$$\mathrm{Im}\; E_n(\lambda) = \mathrm{Im}\; \lambda \int dx\, \left|\psi_n(x)\right|^2 v(x)$$

On B or C

$$\mathrm{Im}\; \lambda \neq 0,$$

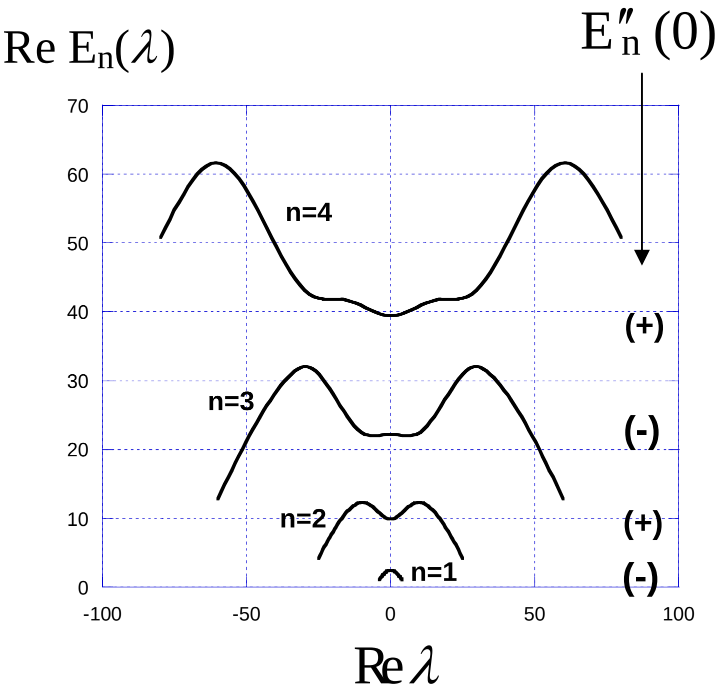
$$\mathrm{Im}\; E_n(\lambda) = 0.$$

Thus on all points of B or C

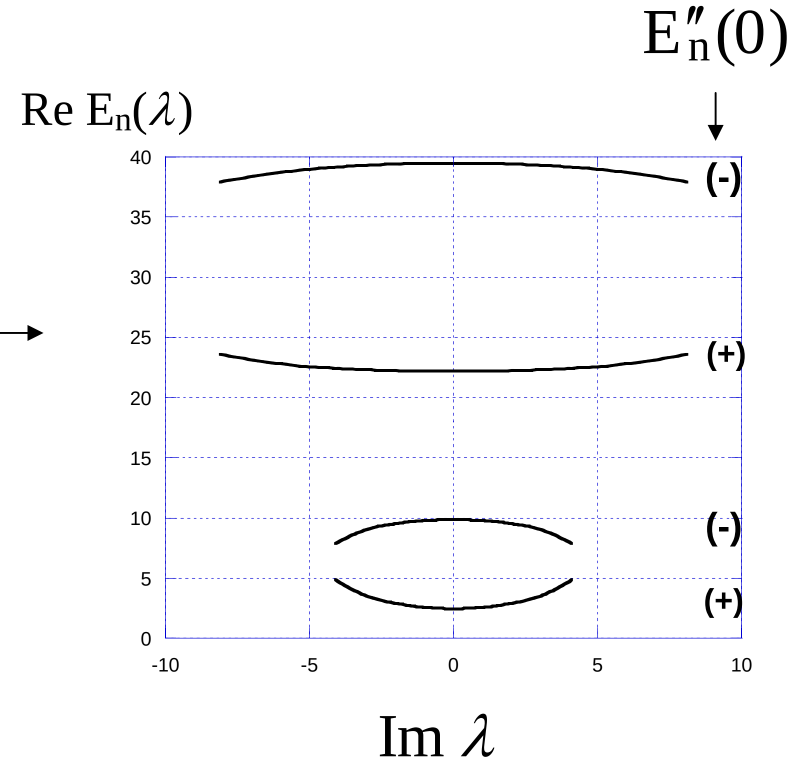
$$\boxed{\int dx\, \left|\psi_n(x)\right|^2 v(x) = 0.}$$

Critical points on the imaginary axis ?

Review: $V(x) = -\lambda\ \mathrm{sgn}(x)$



Laplace's Equation



AVOIDED CROSSINGS FOR REAL COUPLING

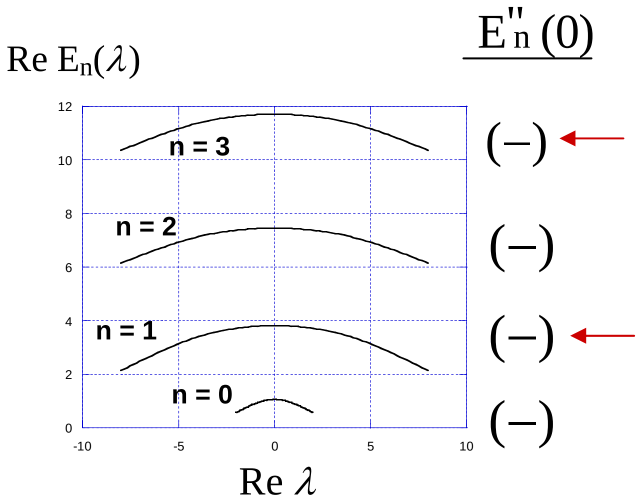
$\Rightarrow$

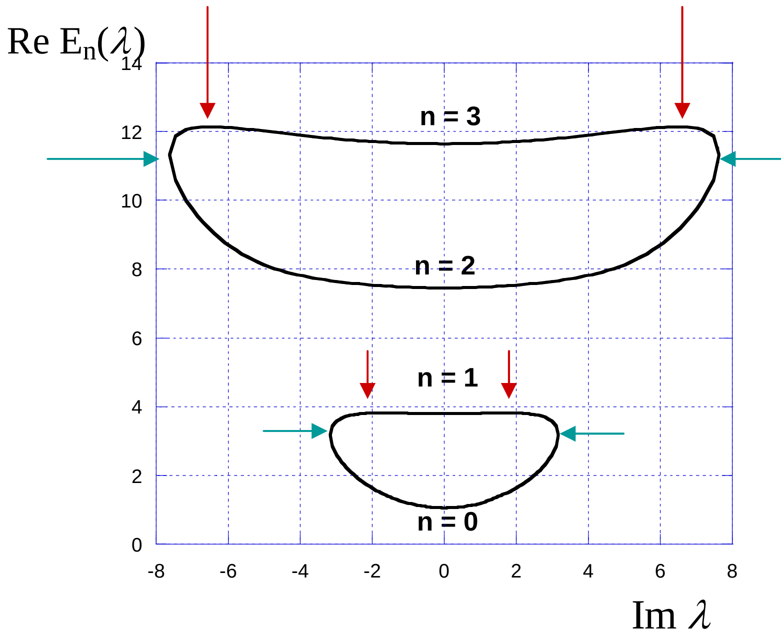
NO NEED FOR CRITICAL POINTS FOR IMAGINARY COUPLING,

IDEAL CASE

$V(x) = \lambda x + x^4$

Bender, Berry, Meisinger, Savage, Simsek (2001)





Sheets n = 1, 3, 5, 7, etc have a critical point before the branch point is reached.

These critical points act as sources of real loci of type C.

BBMSS 2001 J. Phys. A **34** L31

Informal Rule

If $E_n(-\lambda) = E_n(\lambda)$ and the sequence of $E''_n(0)$ is not ideal, there may be critical points on the imaginary axis of λ before any branch point is reached.

These critical points are sources of real loci of type C.

Mathieu's Equation – Quantum Pendulum

New Notation:

$\lambda \to 2q$

$E_n(\lambda) \to a_m(q),\ m = 0, 2, 4, \cdots$

$\to b_m(q),\ m = 2, 4, 6, \cdots$

$$-\psi_m''(x) + 2q\cos(2x)\psi_m(x) = a_m/b_m\,\psi_m(x)$$
$$-\infty \le x \le \infty, \quad \psi_m(x+\pi) = \psi_m(x)$$

Even case:

$\psi_m(-x) = \psi_m(x),\ \ a_m(-q) = a_m(q),\ \ m=0,2,4,\cdots$

Odd case:

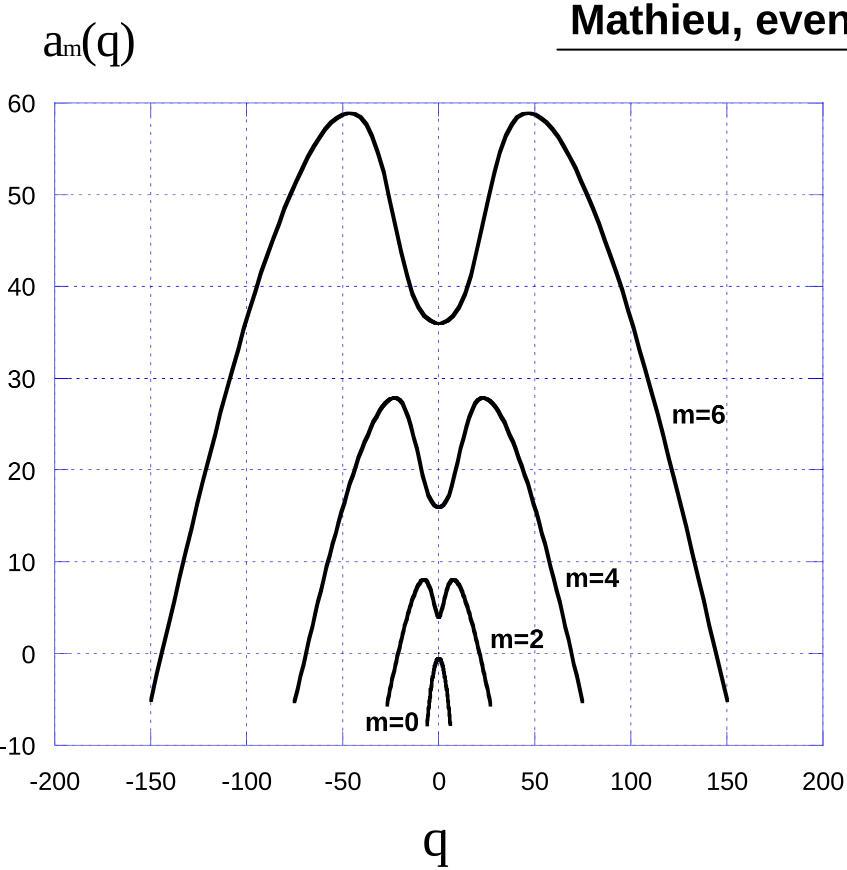
$\psi_m(-x) = -\psi_m(x),\ b_m(-q) = b_m(q),\ m=2,4,6,\cdots$

PT Symmetry:

If $2x \to (\pi - 2x)$ and $q \in i\mathbb{R}$.

$a_m(q)$ and $b_m(q)$ have real loci of type B

Mathieu, even parity, period pi



(1) $a_m(-q) = a_m(q)$

(2) $a_m(q^*) = a_m^*(q)$

(3) m = 0, one critical point

(4) $m \geq 2$, three critical points

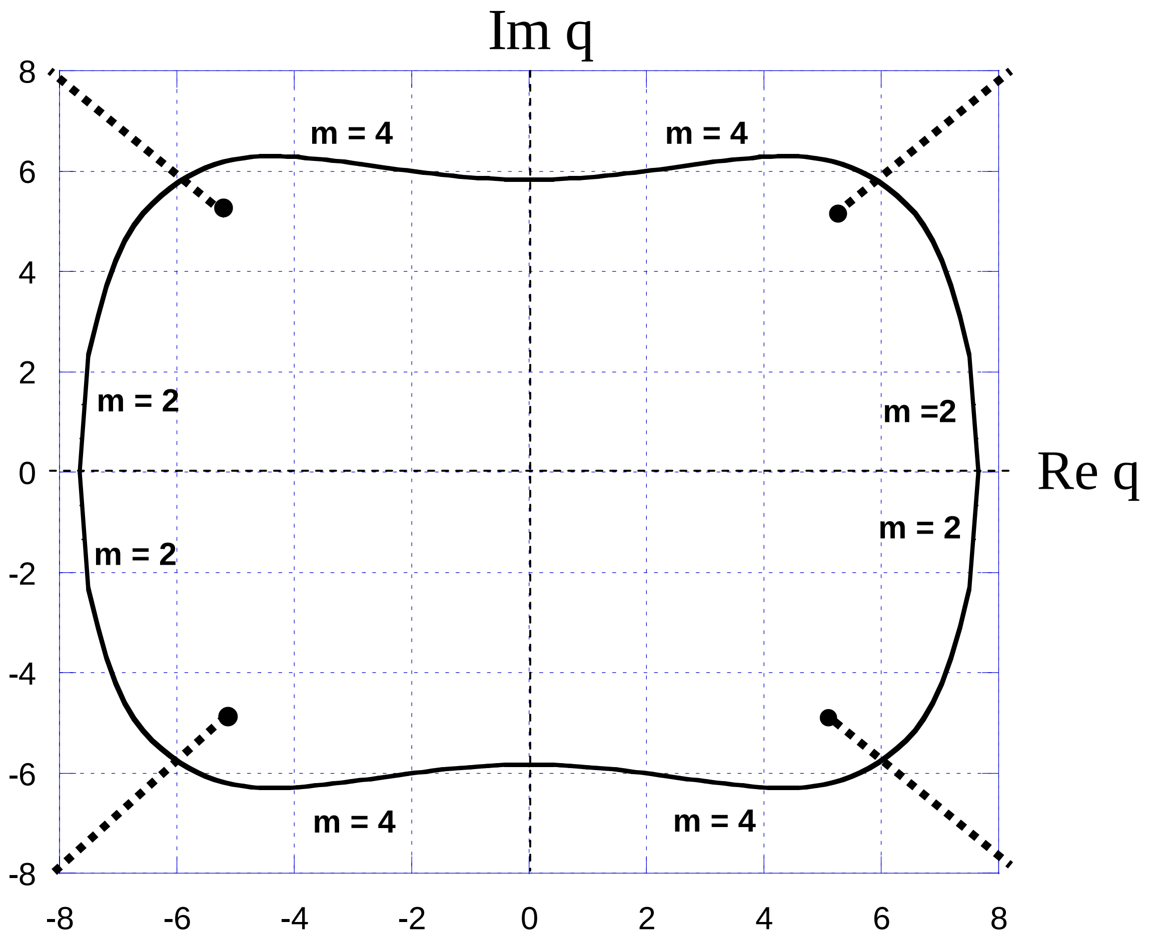
(5) $a''_m(0)$ □ $-, +, +, +, +. \cdots$

(6) 2 cr. pts. on Im q for m=4, 8, 12, ···

Positions of branch points are from

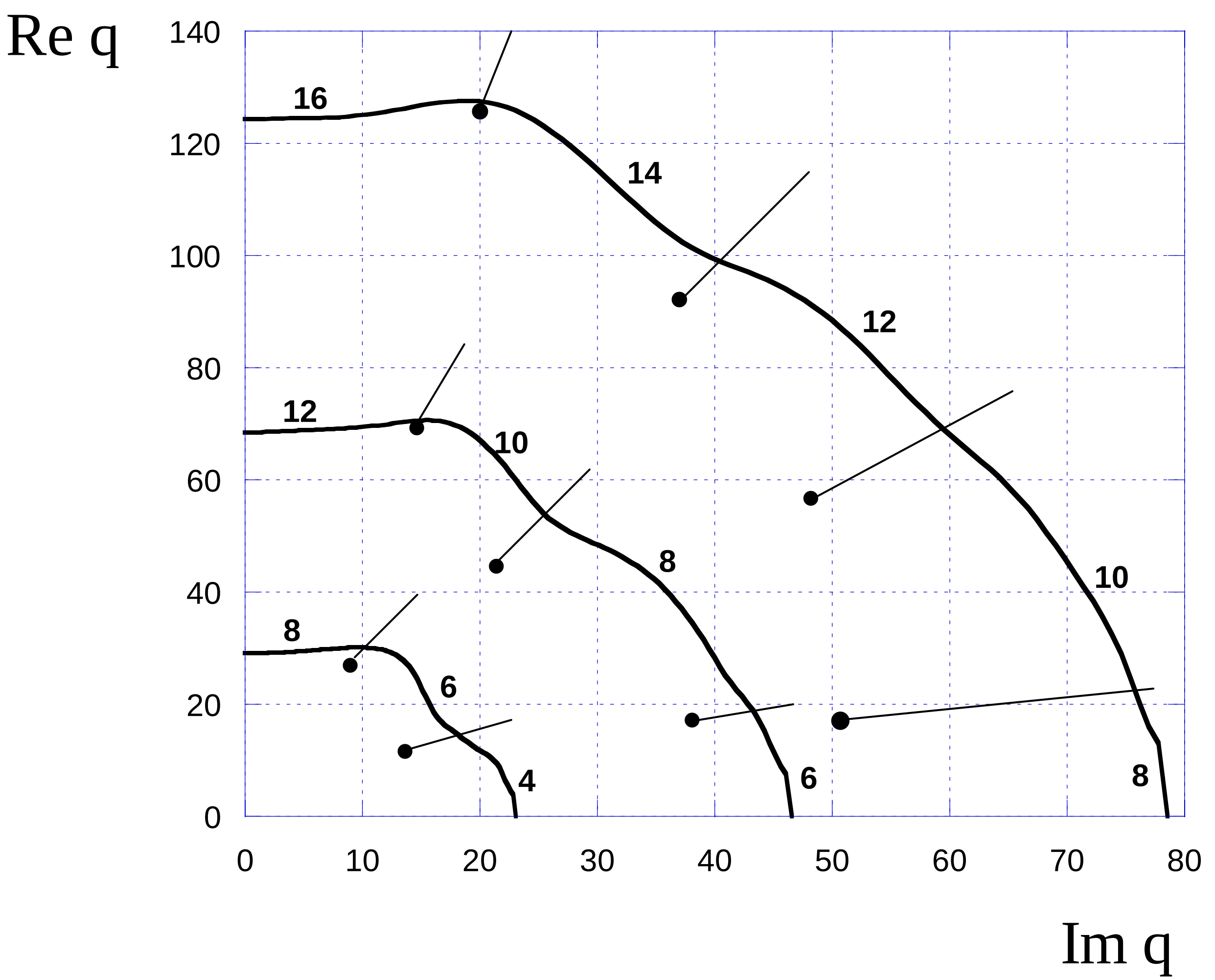
D. Blanch and D. S. Clemm 1980 Math. Comp. **23** 248

Mathieu, even parity, period pi, m = 2 to m = 4

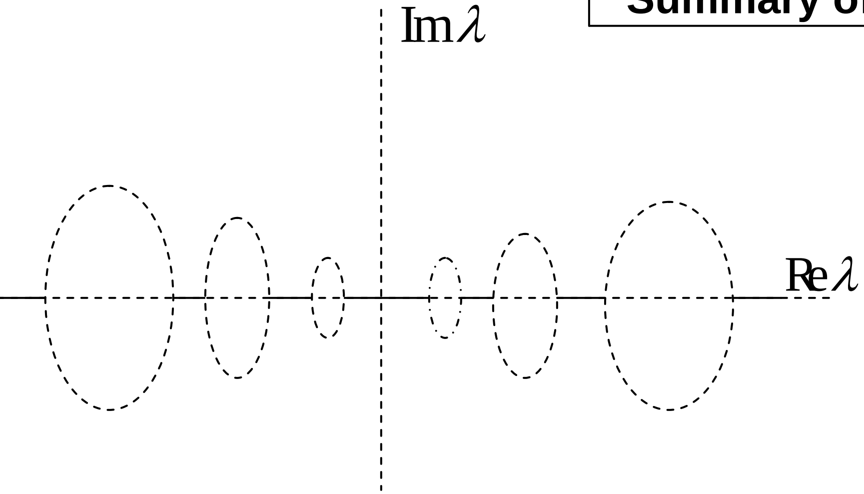


Mathieu Eigenvalue, Even Parity, Period Pi

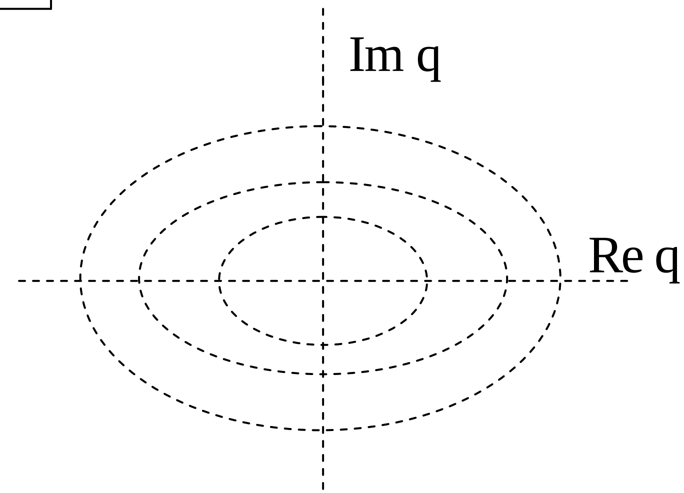
Real Loci Type C



Summary of type C

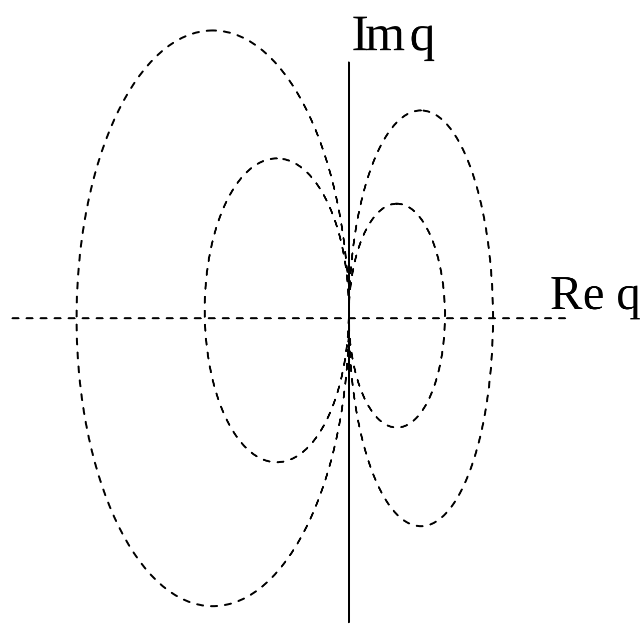


$V(x) = -\lambda \ \mathrm{sgn}(x)$



Mathieu,+,π

Mathieu, +, 2π



$V(x) = \lambda x + x^4$