“Relativity principle” and non-Hermitian dynamics

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Recently it was observed that a non-Hermitian evolution can go faster than the fastest Hermitian evolution

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This conclusion was recently questioned

A. Mostafazadeh, arXiv: 0706.3844v1 [quant-ph] 26 June 2007

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But how to choose a proper Hilbert space?

Does the result depend on the choice of the Hilbert space?

Is it possible to keep working in the usual Hilbert space?

Equivalence transformation

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Let us have a usual N-dimensional Hilbert space $f, g, \psi \ldots \in \mathcal{H}$

$$\langle f|g\rangle = \sum_i f_i^* g_i \qquad f = \sum_i f_i e_i \,, \qquad g = \sum_i g_i e_i \tag{1}$$

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Consider a non-Hermitian $H \neq H^+$ acting in $\mathcal{H}$

$$H\psi_i = E_i\psi_i \qquad \psi_i \in \mathcal{H} \qquad E_i \in \mathbb{R} \tag{2}$$

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In particular H may be $\mathcal{PT}$-symmetric

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This is a condition that $E_i \in \mathbb{R}$

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Two quantum mechanics are unitary equivalent

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a “bad reference frame” (i.e. equivalence class) used

This generalization has a sense only if another generalization is assumed

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being used leads just to the conventional QM description

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Resolution of the identity operator and spectral representation of $H \neq H^+$

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From spectral representation of $\hat{H}$

$$\hat{H} = \sum_i E_i |\varphi_i\rangle\langle\varphi_i| = \mathcal{M}^{-1/2} \sum_i E_i |\psi_i\rangle\langle\psi_i| \mathcal{M}^{-1/2} =: \mathcal{M}^{-1/2} H \mathcal{M}^{1/2} \tag{10}$$

Resolution of the identity operator and spectral representation of $H \neq H^+$

Let us have $H \neq H^+$ $\quad \hat{H} = \mathcal{M}^{-1/2} H \mathcal{M}^{1/2} = \hat{H}^+ \sim H$

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one derives the spectral representation of H

$$H = \sum_i E_i |\psi_i\rangle\langle\psi_i|\mathcal{M}^{-1} \tag{11}$$

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while measuring the observable H (energy in particular)

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Relativity principle gives:

$$p_i = \frac{|\langle\varphi_i|\varphi\rangle|^2}{\langle\varphi_i|\varphi_i\rangle\langle\varphi|\varphi\rangle} = \frac{|\langle\psi_i|\mathcal{M}^{-1}|\psi\rangle|^2}{\langle\psi_i|\mathcal{M}^{-1}|\psi_i\rangle\langle\psi|\mathcal{M}^{-1}|\psi\rangle} \tag{13}$$

$$\varphi = \mathcal{M}^{-1/2}\psi \qquad \varphi_i = \mathcal{M}^{-1/2}\psi_i \qquad \hat{H}\varphi_i = E_i\varphi_i \tag{14}$$

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The closer H is to an exceptional point where $\mathcal{M}$ is singular the more this difference becomes visible

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With the help of $\mathcal{M}$ we can go to the “reference frame” where $H \neq H^+$ and evolution is non-unitary $\varphi = \mathcal{M}^{1/2}\psi$ $\qquad$ $\mathcal{M}$ is time independent

$$i\dot{\psi}(t) = H\psi(t) \qquad \psi(0) = \psi_0 = \mathcal{M}^{-1/2}\varphi_0 \tag{21}$$

$$\hat{U}(t) = \hat{U}^{+}(t) = e^{-i\hat{H}t} = e^{-i\mathcal{M}^{-1/2}H\mathcal{M}^{1/2}t} =: \mathcal{M}^{-1/2}U(t)\mathcal{M}^{1/2} \quad (22)$$

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The choice of a particular element from all equivalent Hamiltonians is similar to placing an observer in either one or another reference frame

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Consider Hamiltonian

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \qquad (31)$$

Bender C M, Brody D C Jones H F and Meister B K 2007 *Phys. Rev. Lett.* **98** 040403

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may become infinitesimal (C. Bender et al.)

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$$|E_+\rangle = \begin{pmatrix} 1 \\ e^{-i\alpha} \end{pmatrix} \qquad |E_-\rangle = \begin{pmatrix} 1 \\ -e^{i\alpha} \end{pmatrix} \quad \sin(\alpha) = \frac{r}{s}\sin(\theta) \qquad (32)$$

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$$U(t) = \frac{e^{-irt\cos\theta}}{\cos\alpha} \begin{pmatrix} \cos(\frac{\omega t}{2} - \alpha) & -i\sin(\frac{\omega t}{2}) \\ -i\sin(\frac{\omega t}{2}) & \cos(\frac{\omega t}{2} + \alpha) \end{pmatrix} \neq U^+(t) \qquad (34)$$

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Hence

These are exceptional points
i.e. the points where Hamiltonian H becomes non-diagonalizable and $\mathcal{M}$ singular

Non-Hermitian evolution of spin

("spin observer reference frame")

We study evolution of spin flip $\boldsymbol{\sigma_z = \sigma_z^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

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$$p_\uparrow = 0,\ p_\downarrow = 1 \quad \Longleftrightarrow \quad \frac{\omega t}{2} - \alpha = \frac{\pi}{2} + N\pi, \quad N = 0, 1, \ldots$$

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Time interval necessary for $|\uparrow\rangle \rightarrow |\downarrow\rangle$:

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Time interval necessary for $|\downarrow\rangle \longrightarrow |\uparrow\rangle$:

$$\Delta t_2 = t_{-1} - t_{+0} = \frac{\pi - 2\alpha}{\omega} = \frac{\pi - 2\alpha}{2s|\cos\alpha|} = \frac{\pi - 2\alpha}{\Delta E} \tag{43}$$

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The closer the Hamiltonian is to a non-diagonalizable matrix
(i.e. $\alpha \longrightarrow -\pi/2$, ΔE fixed)
the more the time interval Δt_1 reduces

Hermitian limit

$$\alpha = 0 \Rightarrow \theta = 0 \qquad H = \begin{pmatrix} r & s \\ s & r \end{pmatrix} = H^{+} \tag{44}$$

$$p_{\downarrow}(t) = \sin^{2}(st) \qquad p_{\uparrow}(t) = \cos^{2}(st)$$

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This means that Hamiltonian (44) realizes an optimal Hermitian evolution between given states.

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For a given eigen-energies difference ΔE
the ratio of non-Hermitian time evolution and the sharpest Hermitian time evolution is

$$\frac{\Delta t_1}{\Delta\tilde{t}} = \frac{\pi + 2\alpha}{\Delta E} : \frac{\pi}{\Delta E} = 1 + \frac{2\alpha}{\pi} \tag{46}$$

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We conclude that for any Hermitian Hamiltonian of type (44) its non-Hermitian deformation towards one EP accelerates the flip of spin from up to down while the deformation towards the other EP decelerates it

Conjecture

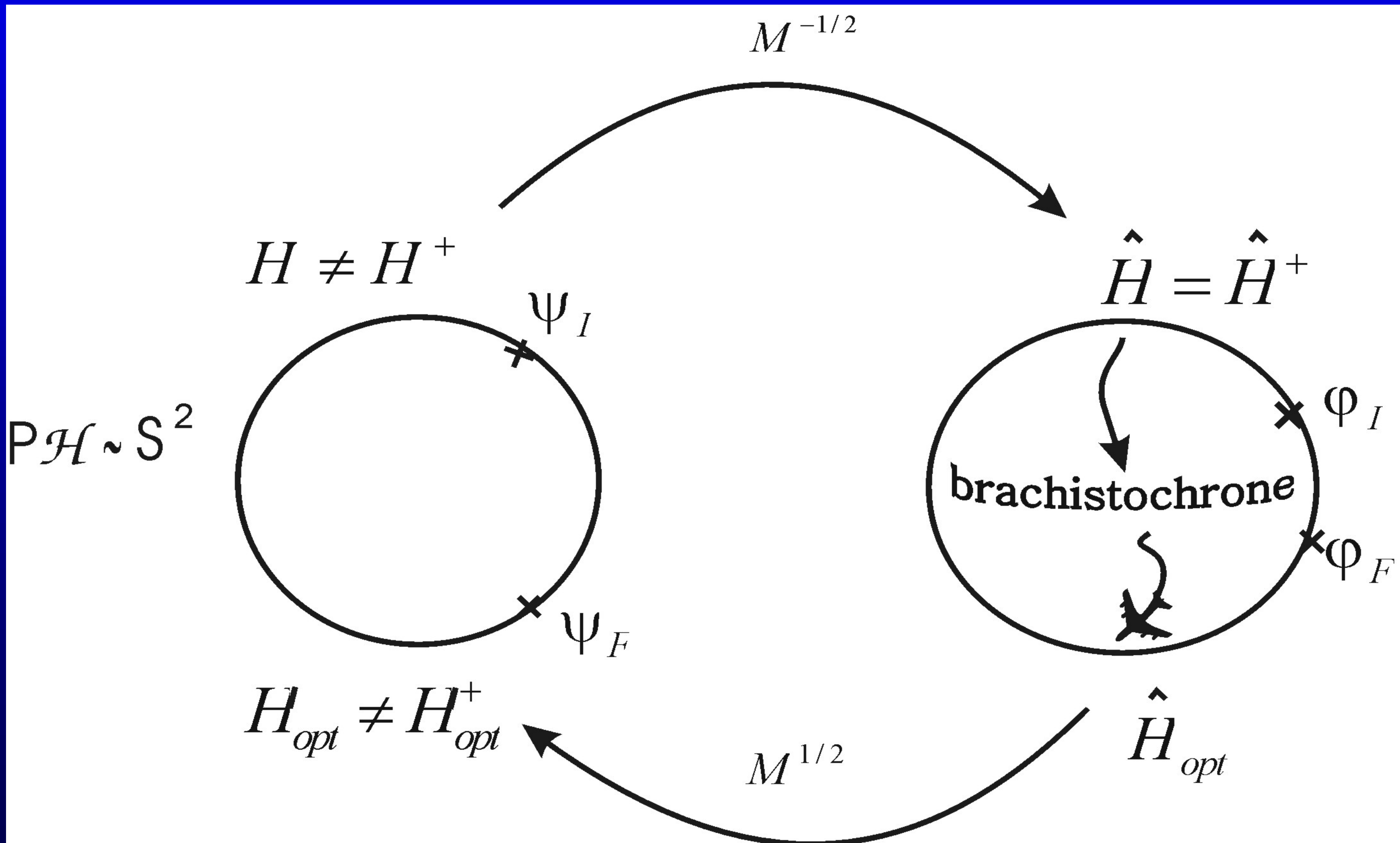
For any physical process described with the help of a Hermitian operator

and any Hermitian Hamiltonian

there exists a non-Hermitian deformation of the Hamiltonian

leading to an acceleration of the process

Non-Hermitian brachistochrone problem



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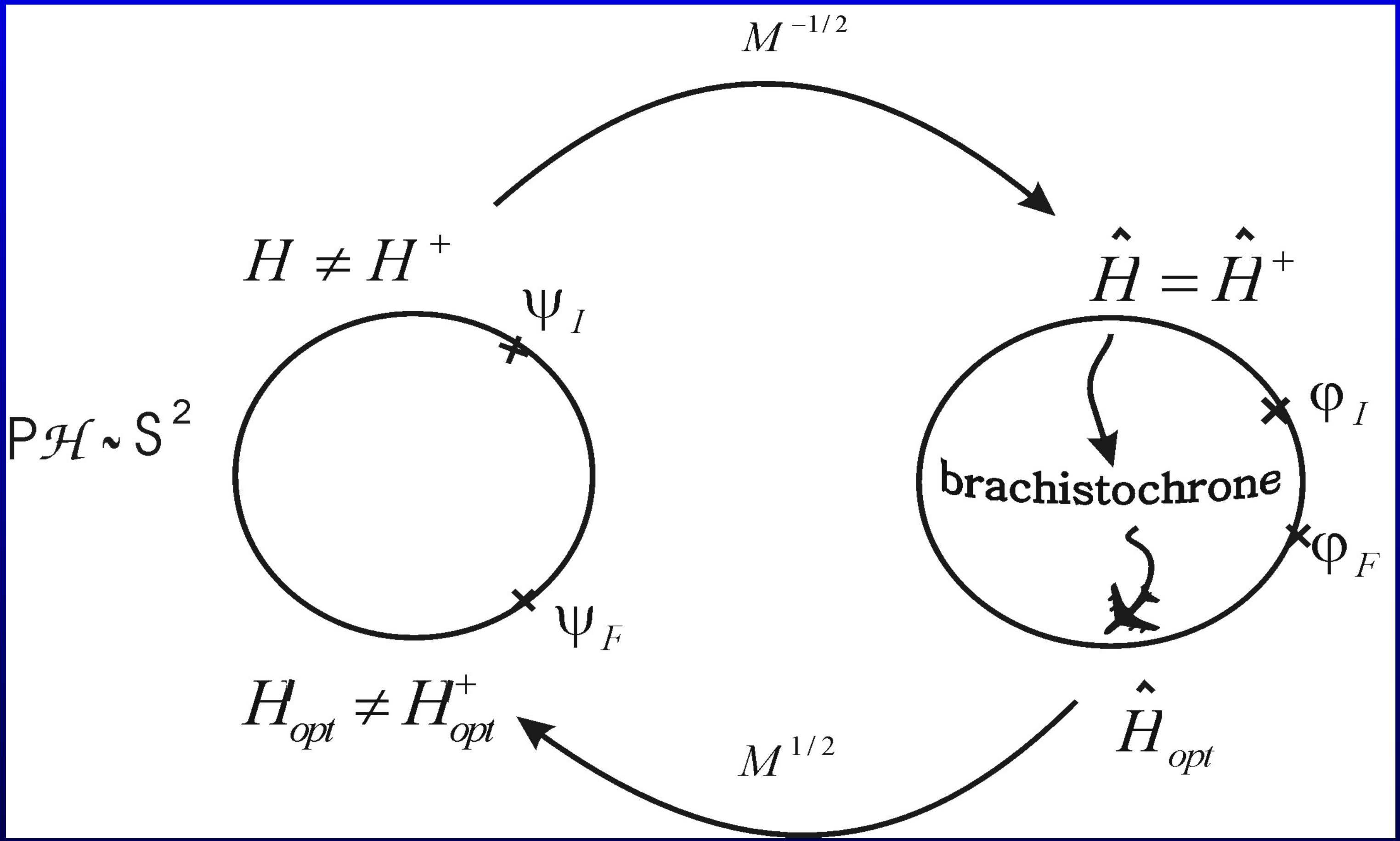

Hamiltonian

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \qquad (47)$$

solves non-Hermitian brachistochrone problem for the states $\psi_I = (1,0)^T$ and $\psi_F = (0,1)^T$

The End