Non-Integer Quantum Numbers of Embryo Orbital Angular Momentum

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Hermitian orbital angular momentum

$$l_{\varphi} = -i\hbar \frac{\partial}{\partial \varphi}$$

Embryo orbital angular momentum

$$\mathrm{L}_{\varphi} = l_{\varphi} - i\eta\hbar\mathrm{T}_2 = -i\hbar\frac{\partial}{\partial\varphi} - i\eta\hbar\mathrm{T}_2$$

T_2 is Non-Hermitian operator

Hermitian orbital angular momentum

$$l_\varphi = -i\hbar \frac{\partial}{\partial \varphi} \qquad m\hbar = 0, \pm\hbar, \pm 2\hbar, \ldots\ldots$$

Embryo orbital angular momentum

$$\mathrm{L}_\varphi = l_\varphi - i\eta\hbar \mathrm{T}_2 = -i\hbar \frac{\partial}{\partial \varphi} - i\eta\hbar \mathrm{T}_2 \qquad \lambda = m\hbar \pm 2m_0\hbar,$$

$$4m_0 = \kappa - 1, \;\; \kappa^2 = 1 + \eta^2$$

T_2 is Non-Hermitian operator

Agenda

1. From Hermitian operator to Embryo Operator

2. Cohortspin $\vec{T}$, (Embryo Operator)

   Non-Hermitian spin angular momentum

3. Embryo orbital angular momentum Operator $\vec{L}$

4. Embryo Special Unitary Group SO(3)

1. From Hermitian operator to Embryo Operator

1.1 $Z^{+} \Rightarrow Z^{\oplus}, \quad Z^{\oplus} = \alpha^{-1} Z^{+} \alpha$

$\oplus \leftarrow\rightarrow \#$

Symbol # was introduced by Ali Mostafazadeh JMP 43, 205-214 (2002)

α *is Matrix Coefficient,*

is Hermitian function or Hermitian matrix

1.2 define Embryo operator : $\mathbf{Z} = \frac{1}{2}(Z^{\oplus} + Z)$

Z : matrix or derivative

1.3

define Embryo operator : $\mathbf{Z}=\frac{1}{2}(Z^{\oplus}+Z)$

$Z:\ matrix$ $\qquad$ $Z\ :derivative$

$$\mathbf{Z}=\frac{1}{2}(\alpha^{-1}Z^{+}\alpha+Z)$$

$$\mathrm{P}_{\xi}=\frac{1}{2}\{\,(-i\frac{\partial}{\partial\xi})^{\oplus}-i\frac{\partial}{\partial\xi}\,\}$$

$$(\frac{\partial}{\partial\xi})^{\oplus}=-\frac{\partial}{\partial\xi}-\alpha^{-1}\frac{\partial}{\partial\xi}\alpha$$

1.1 $Z^{+} \Rightarrow Z^{\oplus}, \; Z^{\oplus} = \alpha^{-1} Z^{+} \alpha$

spinor representation, vector representation

$$\alpha = \mathrm{h}(\varphi) = \begin{bmatrix} \kappa & \eta e^{-i\varphi} \\ \eta e^{+i\varphi} & \kappa \end{bmatrix},$$

$$\alpha = \mathrm{H}(\varphi) = \begin{bmatrix} \kappa^2 & \sqrt{2}\eta\kappa e^{-i\varphi} & \eta^2 e^{-2i\varphi} \\ \sqrt{2}\eta\kappa e^{+i\varphi} & \kappa^2 + \eta^2 & \sqrt{2}\eta\kappa e^{-i\varphi} \\ \eta^2 e^{+2i\varphi} & \sqrt{2}\eta\kappa e^{+i\varphi} & \kappa^2 \end{bmatrix}$$

$$\kappa^2 - \eta^2 = 1$$

2. Cohortspin $\vec{T}$, (Embryo Operator)

Non-Hermitian spin angular momentum

2.1 Spinor representation

$$T_1 = \frac{1}{2}\begin{bmatrix} 0 & e^{-i\varphi} \\ e^{+i\varphi} & 0 \end{bmatrix},\quad T_2 = \frac{1}{2}\begin{bmatrix} -i\eta & -i\kappa e^{-i\varphi} \\ +i\kappa e^{+i\varphi} & +i\eta \end{bmatrix},\quad T_3 = \frac{1}{2}\begin{bmatrix} +\kappa & +\eta e^{-i\varphi} \\ -\eta e^{+i\varphi} & -\kappa \end{bmatrix}$$

$(\vec{T})^{\oplus} = \vec{T}$, Embryo operator, Called Cohortspin

T_2, T_3 are non - Hermitian operators,

Satisfies commutation rule $\vec{T} \times \vec{T} = i\vec{T}$

$$\text{as } \eta = 0, \kappa = \sqrt{1+\eta^2} = 1,$$

$\vec{T}$ back to the combinations of Hermitian operator $\vec{\sigma}$

$$T_1 = \tfrac{1}{2}(\tau_1 \cos\varphi + \tau_2 \sin\varphi)$$

$$T_2 = \tfrac{1}{2}(\tau_2 \cos\varphi - \tau_1 \sin\varphi)$$

$$T_3 = \tfrac{1}{2}\tau_3$$

$$(\vec{T})^+ = \vec{T}, \text{ Hermitian operator}$$

2.2 The route to Cohortspin $\vec{T}$ (I)

2.2.1. $\vec{S} = \frac{1}{2}\vec{\tau} \Rightarrow \vec{\Phi}$, orthogonal real transformation

$$\begin{bmatrix} \mathrm{Cos}\varphi & \mathrm{Sin}\varphi & 0 \\ -\mathrm{Sin}\varphi & \mathrm{Cos}\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}$$

2.2.2. $\vec{\Phi} \Rightarrow \vec{T}$, orthogonal complex transformation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \kappa & -i\eta \\ 0 & +i\eta & \kappa \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

2.3 The route to Cohortspin $\vec{\mathrm{T}}$ (II)

2.3.1 Pauli $\vec{\tau} \Rightarrow (\vec{\tau})^{\oplus} = \mathrm{h}^{-1}\vec{\tau}\,\mathrm{h}, \quad \mathrm{h} = \kappa + 2\eta \mathrm{T}_1$

$$\tau_1^{\oplus} = \tau_1 + 4i\eta\,\mathrm{Sin}\varphi\,\mathrm{T}_3$$

$$\tau_2^{\oplus} = \tau_2 - 4i\eta\,\mathrm{Cos}\varphi\,\mathrm{T}_3$$

$$\tau_3^{\oplus} = \tau_3 + 4i\eta\,\mathrm{T}_2$$

be substituted in following expressions

$$\mathrm{T}_1^{\oplus} = \mathrm{T}_1 + \underline{\frac{1}{2}\{\mathrm{Cos}\varphi(\tau_1^{\oplus} - \tau_1) + \mathrm{Sin}\varphi(\tau_2^{\oplus} - \tau_2)\} = 0}$$

$$\mathrm{T}_2^{\oplus} = \mathrm{T}_2 + \underline{\frac{1}{2}\kappa\{\mathrm{Cos}\varphi(\tau_2^{\oplus} - \tau_2) - \mathrm{Sin}\varphi(\tau_1^{\oplus} - \tau_1)\} + i\frac{1}{2}\eta(\tau_3^{\oplus} + \tau_3) = 0}$$

$$\mathrm{T}_3^{\oplus} = \mathrm{T}_3 \underline{- i\frac{1}{2}\eta\{\mathrm{Cos}\varphi(\tau_2^{\oplus} + \tau_2) - \mathrm{Sin}\varphi(\tau_1^{\oplus} + \tau_1)\} + \frac{1}{2}\kappa(\tau_3^{\oplus} - \tau_3) = 0}$$

3. Embryo orbital angular momentum Operator $\vec{L}$

$$3.1 \qquad \mathrm{L}_{\varphi} \equiv \frac{1}{2}(l_{\varphi}^{\oplus} + l_{\varphi})$$

$$(-i\hbar\frac{\partial}{\partial\varphi})^{\oplus} = -i\hbar\frac{\partial}{\partial\varphi} + \mathrm{h}^{-1}(\varphi)(-i\hbar\frac{\partial}{\partial\varphi})\mathrm{h}(\varphi) \quad \Leftarrow \quad \mathrm{h}(\varphi) = \kappa + 2\eta \mathrm{T}_1$$

$$= -i\hbar\frac{\partial}{\partial\varphi} - 2i\hbar\eta\mathrm{T}_2 \quad = l_{\varphi} - 2i\hbar\eta\mathrm{T}_2$$

$$\mathrm{L}_{\varphi} = -i\hbar\frac{\partial}{\partial\varphi} - i\hbar\eta\mathrm{T}_2 = l_{\varphi} - i\hbar\eta\mathrm{T}_2$$

$$\mathrm{L}_{\varphi}^{\oplus} = (-i\hbar\frac{\partial}{\partial\varphi})^{\oplus} + (-i\hbar\eta\mathrm{T}_2)^{\oplus}$$

$$= -i\hbar\frac{\partial}{\partial\varphi} - 2i\hbar\eta\mathrm{T}_2 + i\hbar\eta\mathrm{T}\varphi_2 = -i\hbar\frac{\partial}{\partial\varphi} - i\hbar\eta\mathrm{T}_2 = \mathrm{L}_{\varphi}$$

3.2 Is there Associated Gegenbauer Equation ?

In Euclidean Space, $g(\theta)=\mathrm{Sin}\,\theta$, from Hermitian adjoint operator of

$$(\partial_\theta)^+ = -\partial_\theta - g^{-1}(\theta)\,\partial_\theta\, g = -\partial_\theta - Cot\theta$$

obtain

1. $\Rightarrow (\hat{L})^+ = \hat{L} \equiv \partial_\theta^2 + Cot\theta\,\partial_\theta \equiv -\vec{l}^{\,2}$

Legendre EQ

$$\hat{L}P_l \equiv \{\partial_\theta^2 + Cot\theta\partial_\vartheta\}P_l = -l(l+1)\,P_l = -\vec{l}^{\,2}$$

Compared with Gegenbauer EQ

$$\hat{G}C_{l,m_0} \equiv \{\partial_\theta^2 + (1+4m_0)Cot\,\theta\partial_\theta\}C_{l,m_0} = -l(l+1+4m_0)C_{l,m_0}$$

2. $\Rightarrow (\hat{G})^+ = \partial_\theta^2 + (1-4m_0)Cot\,\theta\,\partial_\theta + 4m_0 \neq \hat{G} \equiv \partial_\theta^2 + (1+4m_0)Cot\,\theta\partial_\theta$

$(\hat{G})^+ \neq \hat{G}$, *however , possesses real values*

3.3 Embryo Oribital Angular Momentum $\vec{L}$

Hermitian orbital angular momentum

$$l_i l_j - l_j l_i = i\varepsilon_{ijk} l_k$$

$$\alpha = g(\theta)h(\varphi) = (\mathrm{Sin}\,\vartheta)^{1+4m_0}(\kappa + 2\eta T_1)$$

$$\vec{L} = \frac{1}{2}\{(\vec{l})^{\oplus} + \vec{l}\} = \frac{1}{2}\{\alpha^{-1}\vec{l}\alpha + \vec{l}\}$$

Embryo orbital angular momentum

$$L_i L_j - L_j L_i = i\varepsilon_{ijk} L_k \qquad i, j = 1, 2, 3$$

$$L_1 = l_1 + i\eta\hbar \mathrm{Cot}\,\theta \mathrm{Cos}\,\varphi T_2 + i2m_0\hbar \mathrm{Cos}\,\theta \mathrm{Sin}\,\varphi$$

$$L_2 = l_2 + i\eta\hbar \mathrm{Cot}\,\theta \mathrm{Sin}\,\varphi T_2 - i2m_0\hbar \mathrm{Cos}\,\theta \mathrm{C}os\,\varphi$$

$$L_3 = -i\hbar\partial_\varphi - i\hbar T_2$$

In non - Euclidean Space $g(\theta) = (Sin\,\theta)^{1+4m_0}$,
from Embryo adjoint operator of ∂_θ

$$(\partial_\theta)^{\oplus} = -\partial_\theta - g^{-1}(\theta)\,\partial_\theta\, g = -\partial_\theta - (1+4m_0)Cot\theta$$

obtain

$$(\hat{G})^{+} = \hat{G}, \quad \hat{G} \equiv \partial_\theta^2 + (1+4m_0)Cot\,\theta\,\partial_\theta$$

Gengenbauer EQ.

$$\hat{G}C_{l,m_0} \equiv \{\partial_\theta^2 + (1+4m_0)\mathrm{Cot}\,\theta\,\partial_\theta\}C_{l,m_0} = -l(l+1+4m_0)C_{l,m_0} \qquad (m=0) \quad (1)$$

$$\Rightarrow \alpha = h(\varphi)g(\theta) = \mathrm{Sin}^{1+4m_0}\theta(\kappa + 2\eta \mathrm{T}_1)$$

$$-\{\partial_\theta^2 + (1+4m_0)\mathrm{Cot}\theta\partial_\theta - (\mathrm{Sin}\theta)^{-1}(\mathrm{L}_\varphi^2 - 4m_0^2) - 2m_0(1+2m_0)\} = \vec{\mathrm{L}}^2$$

$$\underline{\{\partial_\theta^2 + (1+4m_0)\mathrm{Cot}\theta\partial_\theta - (\mathrm{Sin}\theta)^{-1}(m^2 + 4mm_0) - 2m_0(1+2m_0)}\}C_{l,m_0}^m = -\vec{\mathrm{L}}^2 C_{l,m_0}^m$$

$$\underline{-(l+2m_0)(l+2m_0+1)}\; C_{l,m_0}^m = -\vec{\mathrm{L}}^2\; C_{l,m_0}^m$$

Associated Gegenbauer EQ.

$$\{\partial_\theta^2 + (1+4m_0)\mathrm{Cot}\theta\partial_\theta - (\mathrm{Sin}\theta)^{-1}[m^2 + 4mm_0] + l(l+1+4m_0)\}\, C_{l,m_0}^m = 0 \quad (3)$$

Associated Gegendre EQ.

$$\{\partial_\theta^2 + \mathrm{Cot}\,\theta\,\partial_\theta - (\mathrm{Sin}\,\theta)^{-1}m^2 + l(l+1)\}\, P_l^m = 0 \qquad (m_0 = 0) \quad (2)$$

3.4

Embryo Orbital Angular Momentum Sqare Operator

$$(\vec{l})^2 = -\hbar^2\{\partial_\theta^2 + \mathrm{Cot}\,\theta\partial_\theta - (\mathrm{Sin}\,\theta)^{-2} l_\varphi^2\}$$

$$(\vec{l})^2 \Rightarrow l(l+1)$$

$$l = 0, 1, 2, \ldots..$$

$$(\vec{\mathrm{L}})^2 =$$

$$-\hbar^2\{\partial_\theta^2 + (1+4m_0)\mathrm{Cot}\,\theta\partial_\theta - (\mathrm{Sin}\,\theta)^{-2}(\mathrm{L}_\varphi^2 - 4m_0^2) - 2m_0(1+2m_0)\}$$

$$(\vec{\mathrm{L}})^2 \Rightarrow (l+2m_0)(l+2m_0+1)$$

4. Embryo Special Unitary Group SO(3)

Consinder special unitary operator U for which by definition in Euclidean inner product space

$$\left|(\Phi', \Psi')\right| = \left|(U\Phi, U\Psi)\right| = \left|(\Phi, U^{+}U\Psi)\right| = \left|(\Phi, U^{-1}U\Psi)\right| = \left|(\Phi, \Psi)\right|$$

E.Winger showed that the probabilities of the transformed system is exactly the same as that of the original one, if operator satisfies

$$U^{+}U = UU^{+}, \qquad U^{+} = U^{-1}$$

Each typeof unitary U has its conservative operator

For example,

the case of Hermitian special unitary SO(3)

$$R(\varphi) = \mathrm{Exp}\left(1 - \frac{i}{\hbar}\delta\vec{\varphi}\cdot\vec{l}\right)$$

$\vec{l}$ is conservative generator with respect to space rotation,

$\vec{l}$ is generator of $R(\varphi)$

What happen,

if orbital angular momentum

is non – Hermitian operator ?

$$\vec{l} = \vec{r}\times\vec{p} \Rightarrow \vec{r}\times\vec{\mathrm{P}} = \vec{\mathrm{L}}$$

$\vec{\mathrm{P}}$ is non–Hermitian operatoı

For example,

the case of Embryo special unitary SO(3)

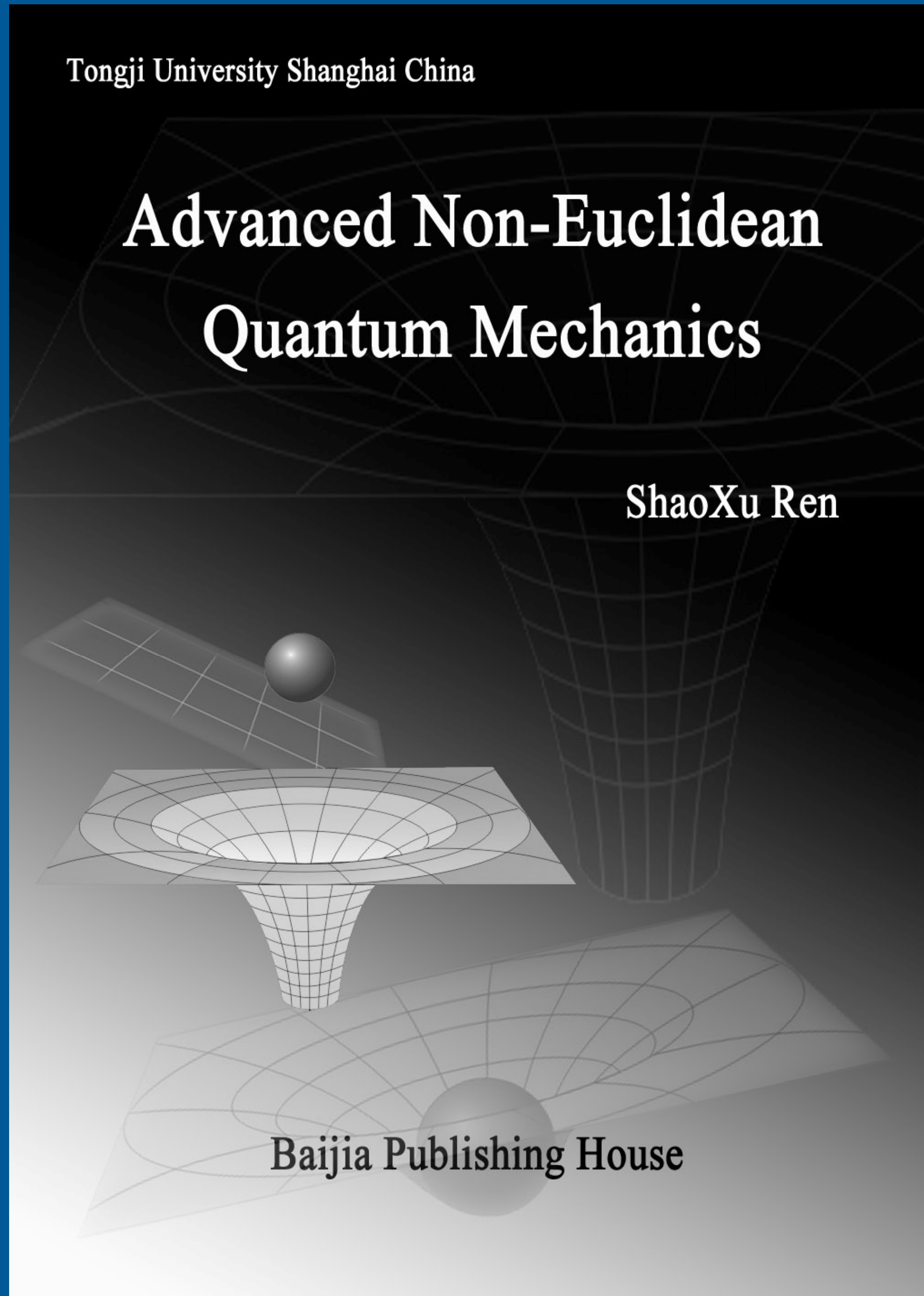
$$\mathrm{R}(\varphi) = \mathrm{Exp}\left(1 - \frac{i}{\hbar}\delta\vec{\varphi}\cdot\vec{\mathrm{L}}\right)$$

$\vec{\mathrm{L}}$ is conservative generator with respect to space rotation,

$\vec{\mathrm{L}}$ is generator of $\mathrm{R}(\varphi)$

$$\vec{l} = \vec{r}\times\vec{p} \Rightarrow \vec{r}\times\vec{\mathrm{P}} = \vec{\mathrm{L}}$$

$\vec{\mathrm{P}}$ and $\vec{\mathrm{L}}$ are non–Hermitian operators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