Quasi–exact oscillators and their tobogganic bound states

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Sturm–Liouville problems

We study polynomial potentials

$$H = -\frac{d^2}{dx^2} + \sum_{k \geq -2} g_k x^k,$$

where $g_{k_{max}} > 0$ and $g_{-2} = \ell(\ell+1) > 0$.

We seek the solution ψ of the Schrödinger equation $H\psi = E\psi$. The asymptotic behaviour is determined by

$$-\frac{d^2\psi(x)}{dx^2} + g_n x^n \psi(x) = 0,$$

where $k_{max} = n$.

$$\begin{aligned} \psi(x) &= C_1 \sqrt{x} I_{\frac{1}{n+2}} \left(\frac{2\sqrt{g_n}}{n+2} x^{\frac{n+2}{2}} \right) + \\ &\quad C_2 \sqrt{x} K_{\frac{1}{n+2}} \left(\frac{2\sqrt{g_n}}{n+2} x^{\frac{n+2}{2}} \right) \end{aligned}$$

Boundary conditions follow from the requirement $\psi \in L^2(\mathcal{C})$. $\mathcal{C}$ used to be $\mathbb{R}$. Asymptotic behaviour for $\xi \to \infty$:

$$I_\nu(\xi) \sim \frac{e^\xi}{\sqrt{2\pi\xi}}$$

$$\psi(x) \sim \exp(\pm\sqrt{g_n}\frac{2}{n+2}x^{\frac{n+2}{2}})$$

$$K_\nu(\xi) \sim \sqrt{\frac{\pi}{2\xi}}e^\xi$$

Bender, Turbiner (1993): the eigenvalue problem can be continued to $\mathbb{C}$.

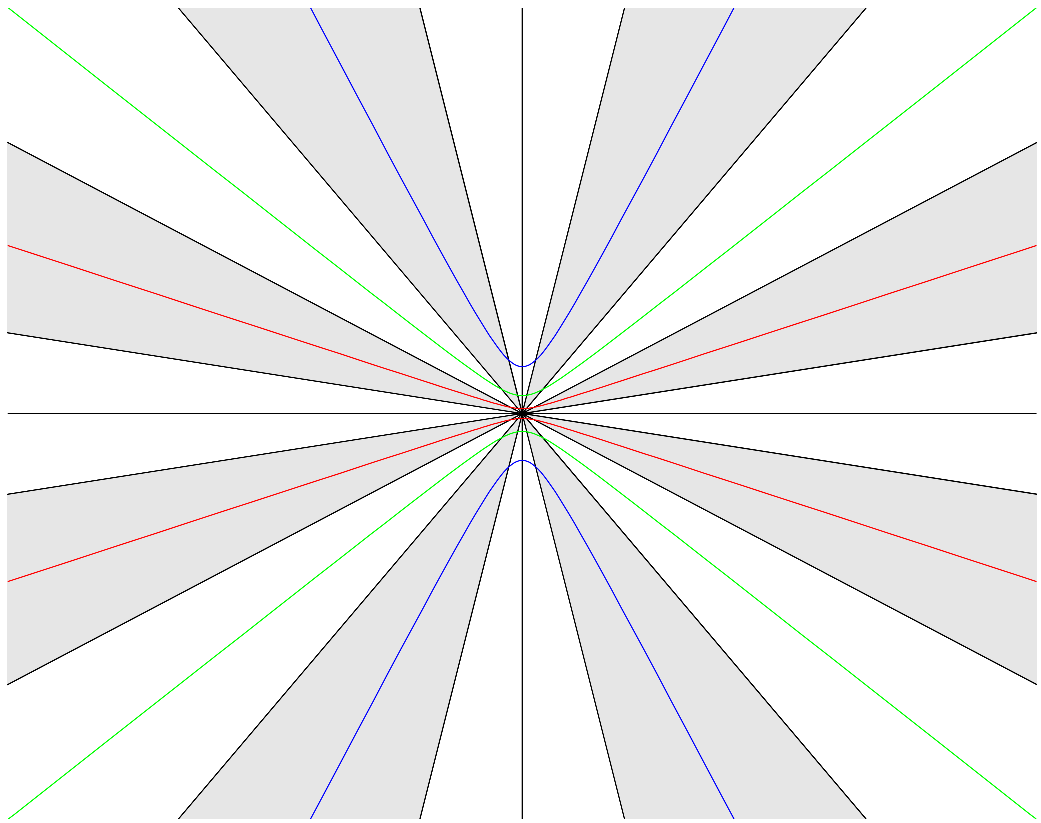
Buslaev, Grecchi (1993) and Bender, Boettcher (1998) showed that $non\text{–}Hermitian$ Hamiltonians can have real spectra when defined along a curve in $\mathbb{C}$.

Along the ray $\varrho\, e^{i\theta}$ we have

$$\psi(\varrho,\varphi) \sim \exp(\tfrac{2}{n+2}\varrho^{\frac{n+2}{2}} \cos \tfrac{n+2}{2}\varphi)$$

i.e. solutions physical $\times$ unphysical (according to the sign of $\cos \frac{n+2}{2}\varphi$). The complex plane is asymptotically divided into Stokes' wedges and we can join them by a path in $\mathbb{C}$.

Imposing suitable boundary conditions on the eigenfunctions in different Stokes' wedges defines different Sturm–Liouville problems generated by the same differential expression.



QES polynomial potentials

For certain sets of potential parameters (a part of) the solution can be found in form $\psi(x) = P_J(x)\exp(-Q_{\frac{n+2}{2}}(x))$, $J > 0$ integer.

Eigenenergies are roots of an algebraic equation $\Rightarrow$ dependence on potential parameters available in a closed form only for low J.

The richest QES on $\mathbb{R}$ is sextic potential:

$$V(x) = x^6 + 2ax^4 + \left(a^2 - (4J-1)\right)x^2$$

Bender, Monou (2005) showed that

$$V(x) = x^6 + 2ax^4 + \left(4J - 1 - a^2\right)x^2$$

exhibits also a QESs when defined along a curve joining asymptotically wedges around $\arg(x) = -45°$ and $\arg(x) = -135°$.

Bender, Boettcher (1998) found that

$$V(x) = -x^4 + 2iax^3 + (a^2 - 2b)x^2 + 2i(ab - J)x$$

defined along a curve joining asymptotically wedges around $\arg(x) = -30^\circ$ and $\arg(x) = -150^\circ$ also has QESs (and a spectrum bounded below).

We see that

- the leading term can be negative and the spectrum real and bounded below
- odd powers are admitted if the corresponding coefficients are imaginary
- there are values of parameters when spectrum is not fully real

Two questions:

- How different are spectra
- Role of the QES

Example:

$$V(x) = x^{14} + ax^{12} + bx^{10} + cx^8 + dx^6 + ex^4 + fx^2$$

$$\begin{aligned} d &= (-448 + 5a^4 - 24a^2b + 16b^2 + 32ac)/64 \\ e &= (-160a - a^5 + 8a^3b - 16ab^2 \\ &\quad -8a^2c + 32bc)/64 \\ f &= (96a^2 + a^6 - 384b - 8a^4b + 16a^2b^2 + \\ &\quad 16a^3c - 64abc + 64c^2)/256 \end{aligned}$$

$$E_0 = (a^3 - 4ab + 8c)/16$$

$$\psi_0(x) = \exp\left(-\frac{x^8}{8} - \frac{ax^6}{12} - \frac{(4b-a^2)x^4}{32} - \frac{(a^3-4ab+8c)x^2}{32}\right)$$

$$V(x) = x^{14}+ax^{12}+bx^{10}+cx^{8}+dx^{6}+ex^{4}+fx^{2}$$

$$\begin{aligned} d &= (448 + 5a^4 - 24a^2b + 16b^2 + 32ac)/64 \\ e &= (160a - a^5 + 8a^3b - 16ab^2 \\ &\quad -8a^2c + 32bc)/64 \\ f &= (-96a^2 + a^6 + 384b - 8a^4b + 16a^2b^2 + \\ &\quad 16a^3c - 64abc + 64c^2)/256 \end{aligned}$$

$$E_0 = (-a^3 + 4ab - 8c)/16$$

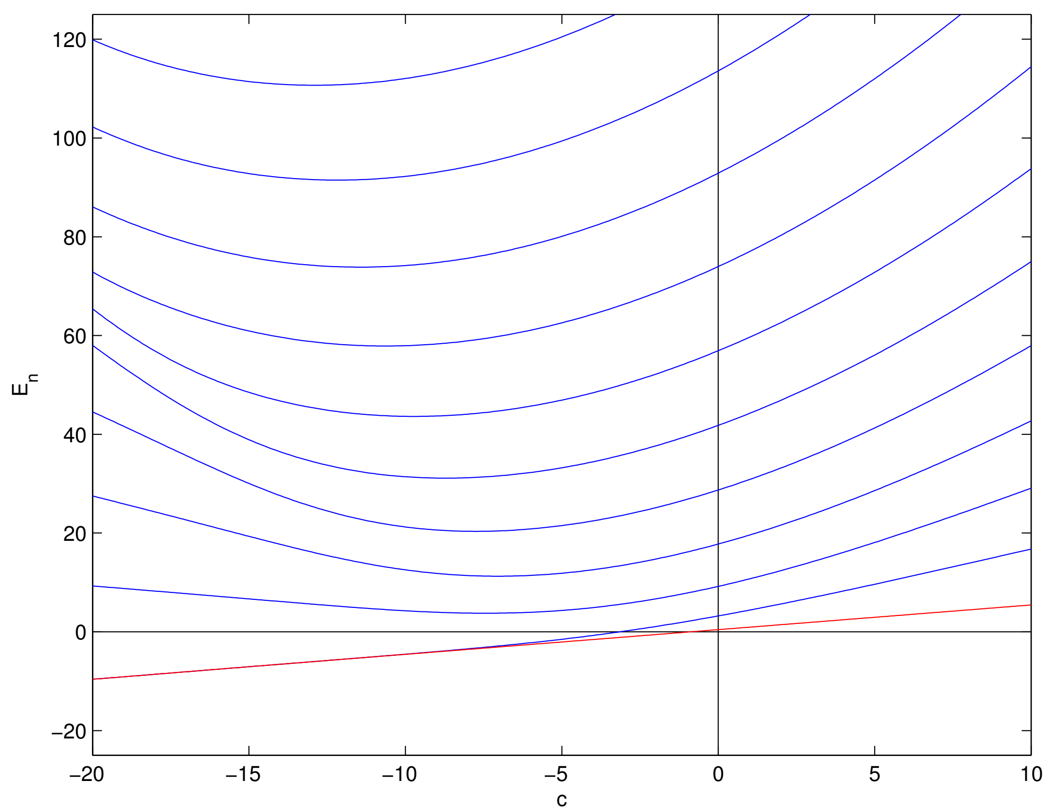
$$\psi_0(x) = \exp\left(\frac{x^8}{8} + \frac{ax^6}{12} + \frac{(4b-a^2)x^4}{32} + \frac{(a^3-4ab+8c)x^2}{32}\right)$$

E_0 is real for all a, b, c. However, if

$$\psi(x) = (1+\alpha x^2)\exp\left(\frac{x^8}{8} + \frac{ax^6}{12} + \frac{(4b-a^2)x^4}{32} + \frac{(a^3-4ab+8c)x^2}{32}\right),$$

then

$$2\alpha^4 + \left(-\frac{a^3}{4} + ab - 2c\right)\alpha^3 + \left(2b - \frac{a^2}{2}\right)\alpha^2 - 2a\alpha + 4 = 0$$



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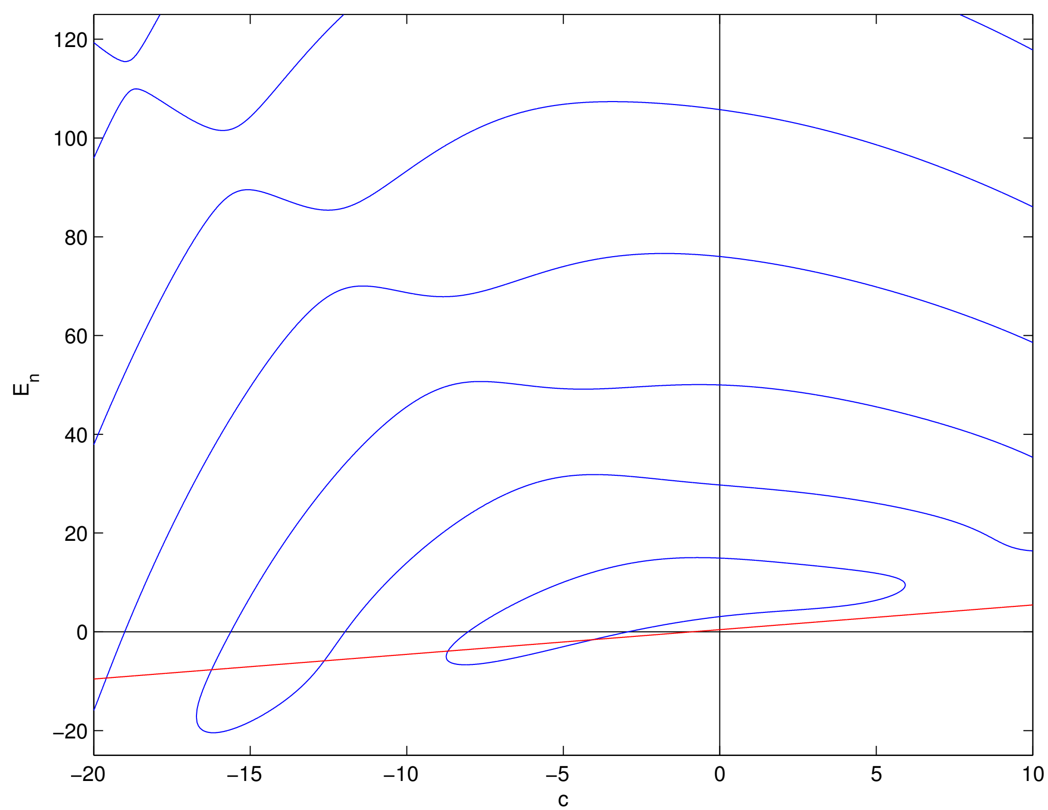

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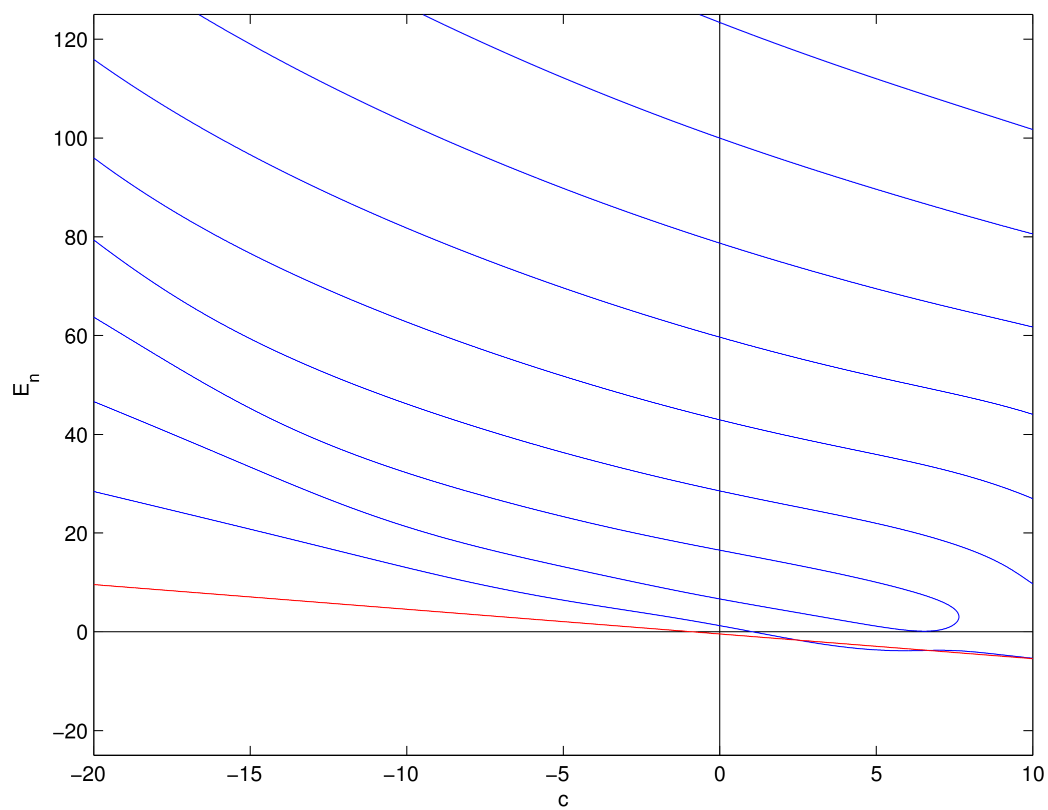

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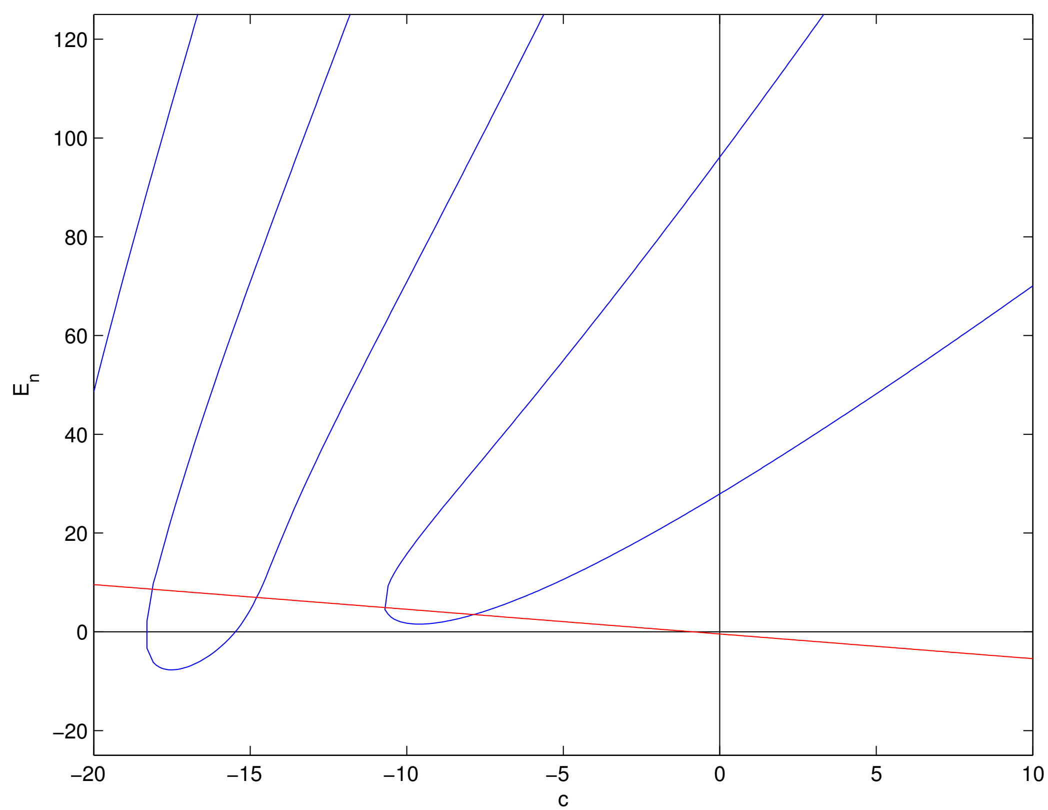

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Eigenvalue problems of the form

$$-\frac{d^2\psi(x)}{dx^2} - (ix)^2 + V(ix)\psi(x) = E\psi(x)$$

can be treated by change of variables (Znojil 2005):

$$ix = (iy)^\alpha \qquad \psi(x) = y^\varrho \varphi(y)$$

If $\varrho = \frac{\alpha-1}{2}$ is chosen, $\varphi^{'}(y)$ is dropped out and

$$0 = -\frac{d^2}{dy^2}\varphi(y) + \frac{\alpha^2-1}{4y^2}\varphi(y) +$$

$$(iy)^{2\alpha-2}\alpha^2\left[-(iy)^{2\alpha} + V\left((iy)^\alpha\right) - E\right]\varphi(y)$$