Perturbation theory for self-adjoint operators in Krein spaces

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TU Berlin

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Topics

1. Operators in Krein spaces
   - $\mathcal{PT}$-symmetric operators
   - Sign types of the spectrum
   - Perturbations
2. Indefinite Sturm-Liouville problems
   - Setting
   - Location of the spectrum

Operators in Krein spaces

$\mathcal{PT}$-symmetric operators are often of the form

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$$

with a $\mathcal{PT}$-symmetric potential V,

$$V(x) = \overline{V(-x)}.$$

such that H is a self-adjoint operator with respect to

$$[f,g] := \int f(x)\overline{g(-x)}\,\mathrm{d}x, \quad f,g \in L^2.$$

Hence:

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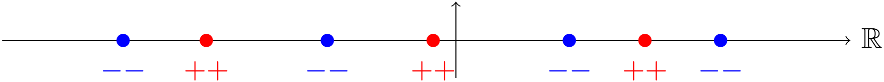
Sign types of the spectrum

Often: Eigenvalues of positive and negative type interlace



an eigenvalue λ belongs to $\sigma_{++}(H)$ if

$$[f, f] > 0 \quad \text{for all eigenfunctions to } \lambda.$$

Stable under some perturbations. Generalize to all spectra:

Definition

$\lambda_0 \in \sigma_{ap}(H)$ if exists $\underbrace{(f_n) \in \operatorname{dom} H \text{ with } \|f_n\| = 1, (H - \lambda)f_n \to 0}_{\text{Approx. Eigensequence}}$.

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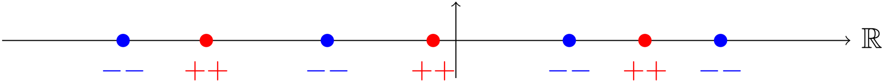

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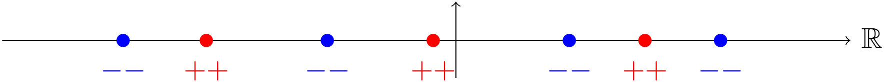

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Similar: $\sigma_{--}(H)$, $\sigma_{\pi_-}(H)$.

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Properties

Theorem (AJT '05)

Let H be a self-adjoint operator in a Krein space $(\mathcal{K}, [\cdot\,,\cdot\,])$ with

$$\sigma_e(H) = \sigma_{++}(H) \cup \sigma_{--}(H).$$

Then

$$\sigma(H) \subset \mathbb{R}.$$

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Theorem 1 [AJT '05]

Let H_0, H_1 be self-adjoint operators in a Krein space with

$$(H_0 - \lambda)^{-1} - (H_1 - \lambda)^{-1} \quad \text{is compact.}$$

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Perturbations small in gap

The gap between two subspaces M and N is defined by

$$\hat{\delta}(M,N) = \|P_M - P_N\|.$$

Theorem 2 [ABJT '08]

Let H_0, H_1 be self-adjoint operators in a Krein space. Let $F \subset \mathbb{C} \cup \{\infty\}$ compact with

$$F \subset \sigma_{++}(H_0) \cup \rho(H_0).$$

Exists $\gamma > 0$ such that for all H_1 with $\hat{\delta}(\operatorname{graph} H_0, \operatorname{graph} H_1) < \gamma$:

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Indefinite Sturm-Liouville problems

We consider:

$$Hy(x) = (\operatorname{sgn} x)(-y''(x) + q(x)y(x)), \qquad \operatorname{dom} H = \mathcal{D}_{\max},$$

where $x \in \mathbb{R}$, $q \in L^1_{\text{loc}}(\mathbb{R})$ real and LP at $\pm\infty$.

NOT self-adjoint with respect to L^2 Hilbert space inner product.
BUTself-adjoint with respect to L^2 Krein space inner product:

$$[f, g] := \int_{\mathbb{R}} f(x)\overline{g(x)} \operatorname{sgn} x \, dx, \quad f, g \in L^2(\mathbb{R}).$$

Questions:

- Spectrum and essential spectrum,
- Accumulation of non-real eigenvalues.

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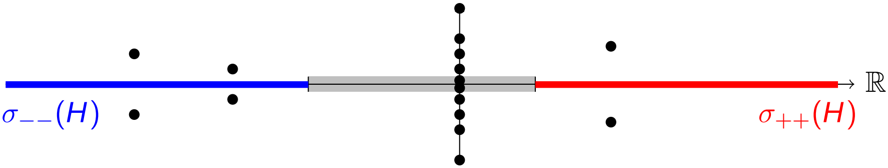
Non-real accumulation



Theorem (KT '07)

There exists a potential q, such that $(i\epsilon_k) \subset \sigma(H)$, $\epsilon_k \to 0$.

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Theorem (KM '07, KT '07)

Then $\rho(H) \neq \emptyset$. H and $H_- \times H_+$ differ by one dimension and

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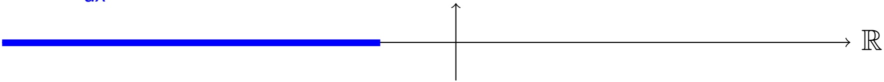

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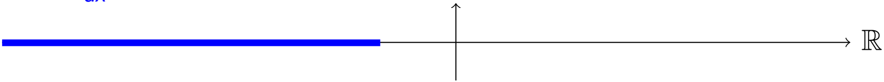

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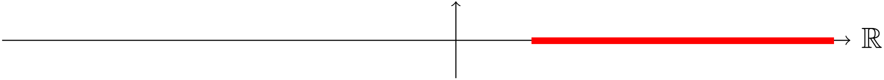

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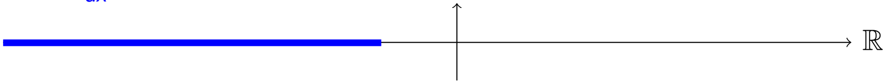

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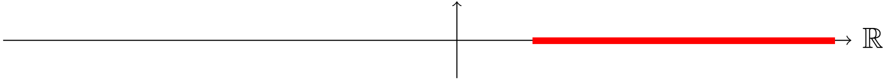

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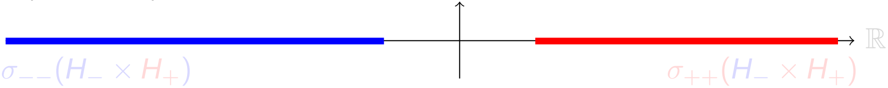
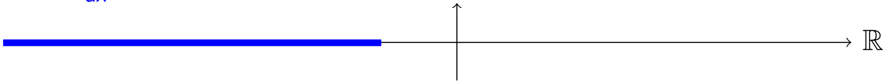
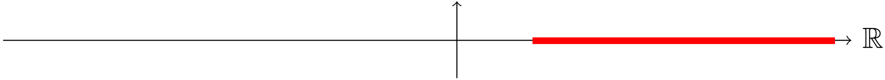


Assume

$H_- = \frac{d^2}{dx^2} - q_- \upharpoonright \{y_-(0) = 0\}, \quad \sigma(H_-)$:



and $H_+ = -\frac{d^2}{dx^2} + q_+ \upharpoonright \{y_+(0) = 0\}, \quad \sigma(H_+)$:



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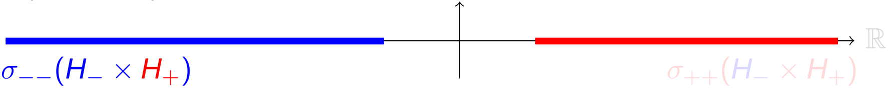

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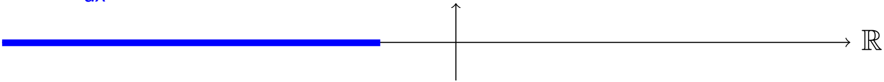

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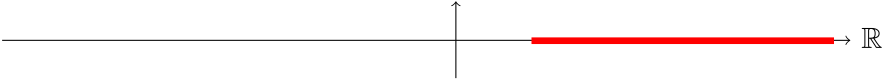

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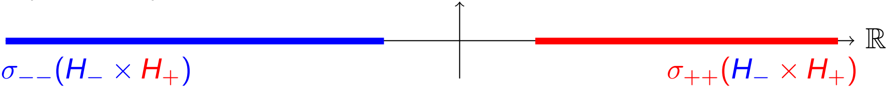
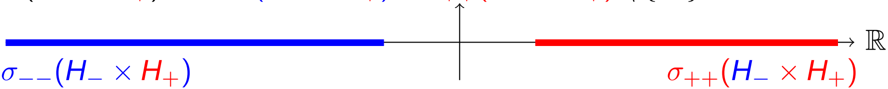


Spectrum of H

$\sigma(H_- \times H_+) = \sigma_{--}(H_- \times H_+) \cup \sigma_{++}(H_- \times H_+) \setminus \{\infty\}$:



$Hy(x) = (\operatorname{sgn} x)(-y''(x) + q(x)y(x))$. H and $H_- \times H_+$ differ by one dimension.

Theorem (Spectrum of H:)

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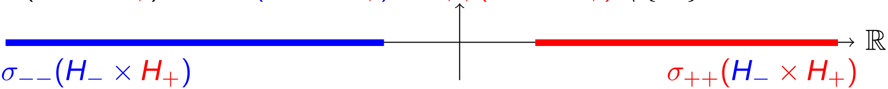

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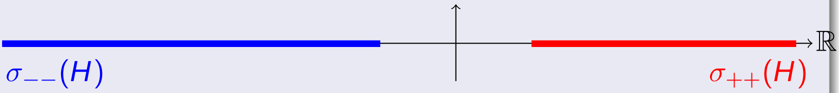

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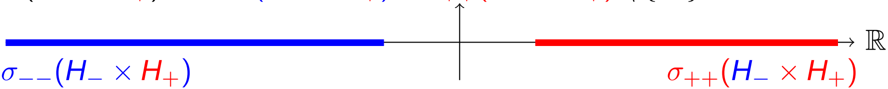

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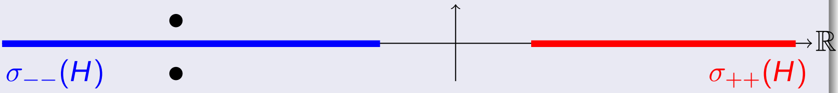

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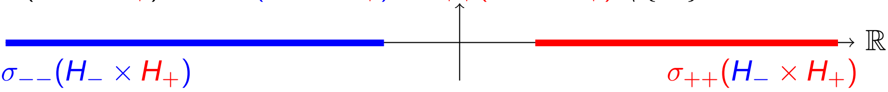

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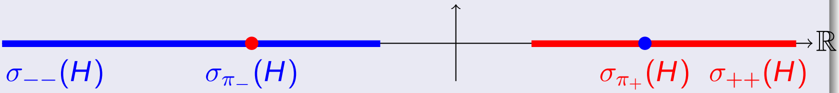

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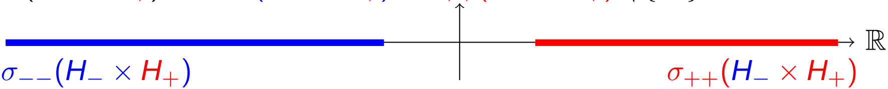

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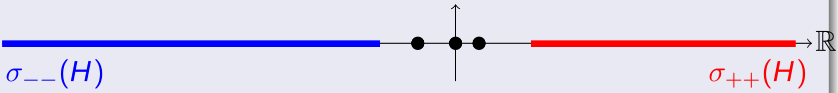
Theorem (Spectrum of H:)

CASE 4 [BMT]: *Up to* *three* *eigenvalues in the gap*



Theorem (BT '07, JDE)

Let H_- and H_+ be as above. Let $m_\pm$ be the classical Titchmarsh-Weyl functions of H_+ and H_-. Set

$$M = m_+ - m_-.$$

Let $M(0)$, $M(\infty)$ exist. (Else $M(0) := \infty$, $M(\infty) := -\infty$). Then

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if

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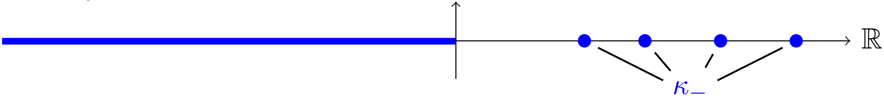

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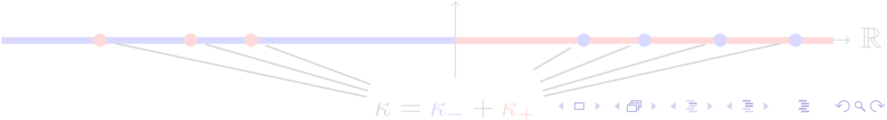
$$\sigma(H_- \times H_+):$$



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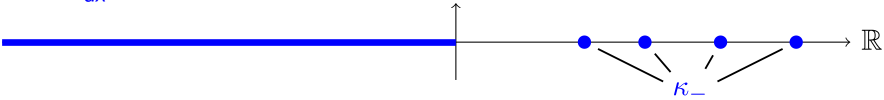

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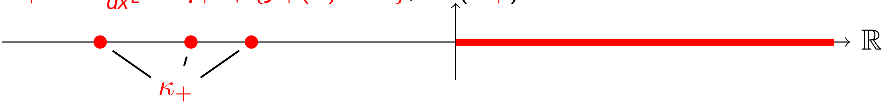

$\sigma(H_- \times H_+)$:

$\kappa = \kappa_- + \kappa_+$

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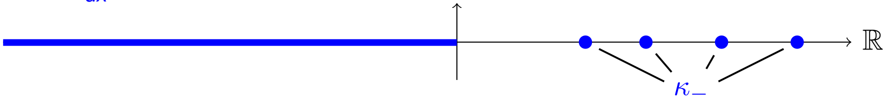

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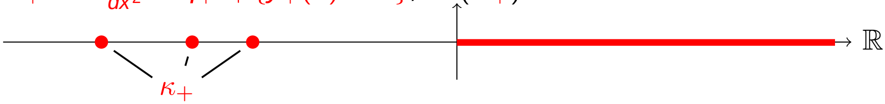

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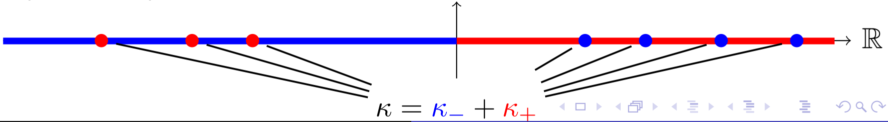

Number of negative squares of H

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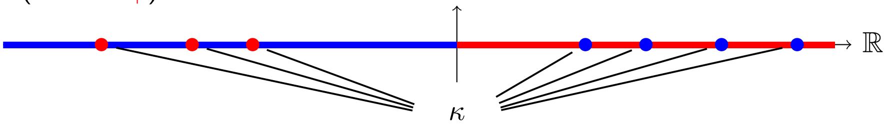

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has finitely many negative squares $\widetilde{\kappa}$. Spectrum $\sigma(H)$:

Number of negative squares of H

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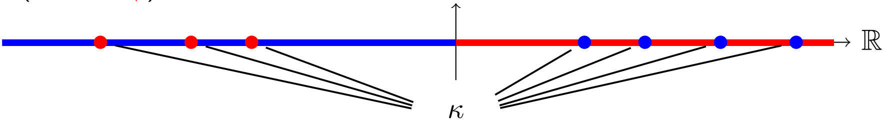

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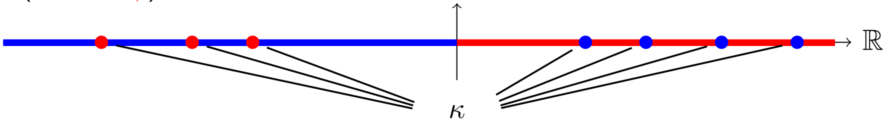

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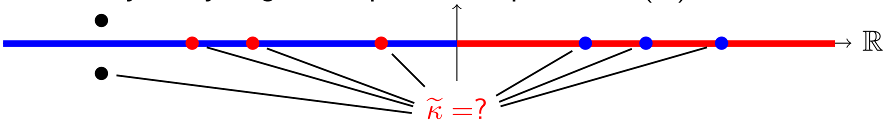

Theorem (BT '07)

Assume, in addition, $\sigma_p(H_-) \cap \sigma_p(H_+) = \emptyset$ *and* $\kappa > 0$. *Let* $m_\pm$ *be the classical Titchmarsh-Weyl functions of* H_+ *and* H_-. *Set*

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$$H \text{ has } \widetilde{\kappa} = \kappa + \Delta_0 + \Delta_\infty \text{ negative squares,}$$

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$$\Delta_0 := \begin{cases} 0, & \text{if } 0 < M(0), \\ -1, & \text{otherwise,} \end{cases} \quad \text{and} \quad \Delta_\infty := \begin{cases} 1, & \text{if } M(\infty) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

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