Dual oscillators with PT symmetry in path integral approach

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The equivalent oscillator (EO) transformation allows to find the correspondence between quantum oscillator systems with attractive and repulsive anharmonic interactions.
In the path integral approach various forms of EO transformation have been considered in:

A. A. Andrianov, Ann.Phys. 140 (1982) 82 ; Phys.Rev. D 76, 025003 (2007)
C. M. Bender, D. C. Brody, J.-H. Chen, H. F. Jones, K. A. Milton, M. C. Ogilvie, Phys. Rev. D 74 (2006) 025016
H. F. Jones, J. Mateo and R. J. Rivers, Phys. Rev. D 74(2006) 125022

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The basic EO-transformation evaluates the partition function for a N-dimensional harmonic oscillator

$$\mathcal{Z}_\beta(\sigma) = \mathrm{Tr}\left[\mathrm{T}\cdot\exp\Big(-\int_0^\beta d\tau \hat{H}(\sigma(\tau))\Big)\right]$$

$$= \int_{\vec{q}(0)=\vec{q}(\beta)} \mathcal{D}^N q(\tau)\ \exp\left\{-\int_0^\beta d\tau \left[\dot{\vec{q}}^{\,2}(\tau) + \tilde{\omega}^2(\tau)\vec{q}^{\,2}(\tau)\right]\right\}, \quad (\vec{q}) = (q_1, \ldots, q_N)$$

with periodic complex frequency

$$\tilde{\omega}^2(\tau) = \omega^2 + i\sigma(\tau); \quad \tilde{\omega}^2(0) = \tilde{\omega}^2(\beta); \quad \beta = 1/kT$$

The real part of the frequency could also depend on Euclidean time

$$\forall\tau,\ \mathrm{Re}\ \tilde{\omega}^2(\tau) = \omega^2 + \nu(\tau) > 0$$

Let's skip it for a while

Formally, after path integration one comes to the functional determinant of the differential operator

$$\mathcal{Z}_\beta(\sigma) = C \parallel -\partial_\tau^2 + \tilde{\omega}^2 \parallel^{-N/2}$$

$$= \left(2\sinh\frac{\omega\beta}{2}\right)^{-N} \exp \mathrm{Tr}\left[\log\left(I + \frac{1}{-\partial_\tau^2 + \omega^2}\; i\sigma(\hat{\tau})\right)\right]$$

Temperature Green function

$$G_\beta(\tau_1 - \tau_2) = \left\langle \tau_1 \left| \frac{1}{-\partial_\tau^2 + \omega^2} \right| \tau_2 \right\rangle = \frac{1}{2\omega \sinh\frac{\omega\beta}{2}} \cosh\left\{\omega\left(\frac{\beta}{2} - |\tau_1 - \tau_2|\right)\right\}$$

Introduce the nonlinear transformation

$$\tilde{\omega}^2(\tau) = \omega^2 + i\sigma(\tau) = -\dot{u}(\tau) + u^2(\tau); \quad u(0) = u(\beta)$$

To provide the one-to-one correspondence $\sigma(\tau) \Longleftrightarrow u(\tau)$

fix the sign $\mathrm{Re}\; u(\tau) > 0$

Transformed action

$$S(\vec{q},u)=\int_0^\beta d\tau\left[\dot{\vec{q}}(\tau)+u(\tau)\vec{q}(\tau)\right]^2$$

change of (periodic) variables

$$\dot{\vec{q}}(\tau)+u(\tau)\vec{q}(\tau)=\dot{\vec{q}}_1(\tau)+\omega\vec{q}_1(\tau)$$

and its Jacobian

$$\left\|\frac{\delta\vec{q}_1}{\delta\vec{q}}\right\|=\left[\frac{\sinh\frac{1}{2}\int_0^\beta d\tau u(\tau)}{\sinh\frac{1}{2}\omega\beta}\right]^N$$

The partition function becomes a local function of

$$\int_0^\beta d\tau u(\tau)$$

$$\mathcal{Z}_\beta(u)=\left(2\sinh\frac{\omega\beta}{2}\right)^{-N}\left\|\frac{\delta\vec{q}}{\delta\vec{q}_1}\right\|=\left[2\sinh\int_0^\beta d\tau\frac{1}{2}u(\tau)\right]^{-N}$$

This is a master formula of the EO-transformation

Consider the spherically symmetric anharmonic oscillator

$$\omega^2 \vec{q}^{\,2} + V_{an}(\vec{q}^{\,2})$$

Legendre transformation from the O(N) symmetric QM to a dual one-dimensional QM with the same energy spectrum

$$\int \frac{\mathcal{D}\sigma(\tau)}{const} \exp\left\{ i \int_0^\beta d\tau \sigma(\tau)\Big(\gamma(\tau) - \vec{q}^{\,2}(\tau)\Big) \right\}$$

Change variables $\sigma(\tau) \Longrightarrow u(\tau)$

Its Jacobian in the integral

$$\delta\sigma(\tau) = i\Big(\partial_\tau - 2u(\tau)\Big)\delta u(\tau); \quad \left\| -i\frac{\delta\sigma}{\delta u} \right\| = \left\| \Big(\partial_\tau - 2u(\tau)\Big) \right\|$$

can be calculated in a similar way,

$$= \tilde{C} \cdot 2\sinh \int_0^\beta d\tau u(\tau)$$

Dual potential systems

$$\mathcal{Z}_\beta = \int\limits_{\vec{q}(0)=\vec{q}(\beta)} \mathcal{D}^N q(\tau)\ \exp\left\{ -\int\limits_0^\beta d\tau \left[\dot{\vec{q}}^{\,2}(\tau) + \omega^2 \vec{q}^{\,2}(\tau) + V_{an}\left(\vec{q}^{\,2}(\tau)\right)\right]\right\}$$

$$= \int\limits_{u,\gamma(0)=u,\gamma(\beta)} \mathcal{D}u(\tau)\mathcal{D}\gamma(\tau)\Phi\left(\int\limits_0^\beta d\tau u(\tau)\right)$$

$$\times \exp\left\{ \int\limits_0^\beta d\tau \left[u(\tau)\dot{\gamma}(\tau) + \gamma(\tau)u^2(\tau) - \omega^2\gamma(\tau) - V_{an}\left(\gamma(\tau)\right)\right]\right\}$$

$$\Phi(z) = \frac{2\sinh z}{[2\sinh(z/2)]^N}$$

Complex "momentum" runs along $\boxed{\mathrm{Re}\ (-\dot{u} + u^2) = \omega^2, \quad \mathrm{Re}\ u > 0}$

Expansion in angular momentum is generated by

$$\Phi(z) = \sum_{l=0}^{\infty} a_l^{(N)} \ \exp\Big(- (\frac{N}{2} + l - 1) z \Big); \quad a_l^{(N)} \equiv (N + 2l - 2) \frac{(N + l - 3)!}{(N-2)! \ l!} \ \text{ for } N > 1;$$

$$\Phi(z) = \exp(\frac{1}{2} z) + \exp(-\frac{1}{2} z) \ \text{ for } N = 1$$

<u>Decomposition of the partition function</u>

$$\mathcal{Z}_\beta = \sum_{l=0}^{\infty} a_l^{(N)} \mathcal{Z}_l(\beta); \qquad \mathcal{Z}_\beta^l = \int\limits_{u,\gamma(0)=u,\gamma(\beta)} \mathcal{D}u(\tau)\mathcal{D}\gamma(\tau)$$

$$\times \exp\Big\{ - \int_0^\beta d\tau \Big[- u(\tau)\dot{\gamma}(\tau) - \gamma(\tau)u^2(\tau) + \omega^2\gamma(\tau) + V_{an}\Big(\gamma(\tau)\Big) + \Big(\frac{N}{2} + l - 1\Big) u(\tau) \Big] \Big\}$$

Special case of quartic anharmonicity

$$V_{an}(\vec{q}^{\,2}) = \lambda(\vec{q}^{\,2})^2$$

integrate in $\gamma(\tau)$

$$S[u,\gamma;\lambda] \Rightarrow S[u;\lambda] = \int_0^\beta d\tau \Big[-\frac{(\dot{u}(\tau))^2}{4\lambda} - \frac{\left(u^2(\tau)-\omega^2\right)^2}{4\lambda} + \Big(\frac{N}{2}+l-1\Big)u(\tau)\Big]$$

"anomaly"

Perturbation theory with stationary configuration along the imaginary axis

$$u(\tau) = \omega + i2\sqrt{\lambda}v(\tau)$$

$$S[v;\lambda] = \int_0^\beta d\tau \Big[(\dot{v}(\tau))^2 - 4\lambda v^4(\tau) + 8i\sqrt{\lambda}\omega v^3(\tau) + 4\omega^2 v^2(\tau) + \Big(\frac{N}{2}+l-1\Big)(\omega + i2\sqrt{\lambda}v(\tau))\Big]$$

The set of anharmonic oscillators with the complex quartic potential unbounded from below describes the same energy spectrum as the original real O(N) symmetric Hamiltonian with the potential bounded from below

Duality for quantum pendulums

with variable radius

$$\mathcal{Z}_\beta(R) = \int\limits_{\vec{q}(0)=\vec{q}(\beta)} \mathcal{D}^N q(\tau) \prod_\tau \delta\Big(\vec{q}^{\,2}(\tau) - R^2(\tau)\Big) \exp\left\{ -\int_0^\beta d\tau \left[\dot{\vec{q}}^{\,2}(\tau)\right] \right\} = \sum_{l=0}^{\infty} a_l^{(N)} \mathcal{Z}_\beta^l(R);$$

$$\mathcal{Z}_\beta^l(R) = \prod_{\tau=0}^{\beta} \frac{R_0}{R(\tau)} \exp\left\{ -\int_0^\beta d\tau \left[(\dot{R}(\tau))^2 + \frac{\Big(N+2l-2\Big)^2}{16R^2(\tau)} \right] \right\}$$

$$\stackrel{R(\tau)=R_0}{\Longrightarrow} \exp\left\{ -\beta \frac{\Big(N+2l-2\Big)^2}{16R_0^2} \right\}.$$

Ultralocal quadratic action admitting the exact calculation of energy spectrum after the contour deformation $u \longrightarrow i\tilde{u}$

Normalization

$$\prod_{\tau=0}^{\beta} \frac{R_0}{R(\tau)} = 1 \longrightarrow \int_0^\beta d\tau \ln \frac{R_0}{R(\tau)} = 0$$

Level splitting in magnetic field

$N = 2; \quad \vec{q} = (q_1, q_2)$

Action

$$S(B) = \int_0^\beta d\tau \left[\dot{\vec{q}}^{\,2}(\tau) + 2i\ B\epsilon_{jk}\ \dot{q}_j q_k + \omega^2(\tau)\vec{q}^{\,2}(\tau) + \lambda\left(\vec{q}^{\,2}\right)^2(\tau) \right]$$

$B \to B(\tau)$

EO transformation **(the Jacobian is the same !)**

$$\lambda\left(\vec{q}^{\,2}\right)^2(\tau) \Longrightarrow i\sigma(\tau)\vec{q}^{\,2} + \frac{\sigma^2}{4\lambda} \qquad \omega^2 + B^2 + i\sigma(\tau) = -\dot{u}(\tau) + u^2(\tau)$$

Angular momentum functional

$$\tilde{\Phi}_2(u, B) = \frac{2\sinh \int_0^\beta d\tau u(\tau)}{4\sinh\left(\frac{1}{2}\left(B\beta + \int_0^\beta d\tau(u(\tau)\right)\right)\sinh\left(\frac{1}{2}\left(-B\beta + \int_0^\beta d\tau(u(\tau)\right)\right)}$$

The spectrum of anharmonic oscillator is split by magnetic field

$$S(u, n) = S(u, 0) + n\int_0^\beta d\tau(u(\tau) \Longrightarrow S(u, n, B) = S(u, 0) + n\int_0^\beta d\tau(u(\tau) + mB\beta$$

Dual form of correlators

Generating functional of external source $\eta_j(\tau)$

$$\mathcal{Z}(\sigma,\eta) = \int \frac{\mathcal{D}^N q(\tau)}{const} \exp\left\{ -\int_0^\beta d\tau \left[\dot{\vec{q}}^{\,2}(\tau) + \tilde{\omega}^2(\tau)\vec{q}^{\,2}(\tau) - \vec{q}\cdot\vec{\eta}(\tau) \right] \right\}$$

$$= C \exp\left\{ \mathrm{Tr}\left[\log(-\partial_\tau^2 + \tilde{\omega}^2)\right] + \frac{1}{4}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \sum_{j=1}^N \eta_j(\tau_1) G_\beta^{(2)}(\tau_1,\tau_2|\sigma)\eta_j(\tau_2) \right\}$$

Correlators

$$\left\langle \hat{q}_{j(1)}(\tau_1)\cdots\hat{q}_{j(2n)}(\tau_{2n}) \right\rangle = \frac{1}{\mathcal{Z}_\beta(0)} \frac{\delta}{\delta\eta_{j(1)}(\tau_1)} \cdots \frac{\delta}{\delta\eta_{j(2n)}(\tau_{2n})} \mathcal{Z}_\beta(\eta)\Big|_{\eta=0}$$

EO-transformation of the Green function

$$G^{(2)}_\beta(\tau_1,\tau_2|\sigma) = \langle\tau|\frac{1}{-\partial_\tau^2+\tilde\omega^2}|\tau_1\rangle \overset{\sigma\to u}{\to} \int_0^\beta d\tau\langle\tau_1|\frac{1}{-\partial_\tau+u(\tau)}|\tau\rangle\langle\tau|\frac{1}{\partial_\tau+u(\tau)}|\tau_2\rangle$$

Zero temperature limit $\beta \to \infty$

$$G^{(2)}_\infty(\tau_1,\tau_2|u) = \int_{-\infty}^{\infty} d\tau\theta(\tau-\tau_1)\theta(\tau-\tau_2)\exp\Big(-\int_{-\infty}^{\infty} d\tau' u(\tau')J(\tau',\tau,\tau_1,\tau_2)\Big)$$

$$J^{(2)}(\tau',\tau,\tau_1,\tau_2) \equiv \theta(\tau-\tau')\big[\theta(\tau'-\tau_1)+\theta(\tau'-\tau_2)\big]$$

``holonomy'' exponential along the contours connecting two points

n-point correlators in dual representation

$$\left\langle \hat{q}_{j(1)}(\tau_1)\cdots\hat{q}_{j(2n)}(\tau_{2n})\right\rangle = \frac{1}{2^n}\prod_{l=1}^{n}\sum_{\{k_l\}\cup\{m_l\}}\int_{-\infty}^{\infty} d\tau_l' \delta_{j(k_l),j(m_l)}\theta(\tau_l'-\tau_{k_l})\theta(\tau_l'-\tau_{m_l})\Xi(\tau,\tau')$$

$$\Xi(\{\tau_k\},\{\tau_l'\}) \equiv \int \frac{\mathcal{D}u(\tau)\mathcal{D}\gamma(\tau)}{const}$$

$$\times \exp\left\{-\int_{-\infty}^{\infty} d\tau\left[-u(\tau)\dot{\gamma}(\tau)-\gamma(\tau)u^2(\tau)+\omega^2\gamma(\tau)+V_{an}\Big(\gamma(\tau)\Big)+\Big(\frac{N}{2}-1\Big)u(\tau)+J^{(2n)}(\tau)u(\tau)\right]\right.$$

Combinatorics

$$\dim\{k_l\}=\dim\{m_l\}=n,\quad \{k_l\}\cup\{m_l\}=\mathrm{Perm}\{1,\cdots,2n\},\ \{k_l\}\cap\{m_l\}=\oslash,$$

External current includes all contours linking triples of points τ_l' **with** τ_{k_l} **and** τ_{m_l}

$$J^{(2n)}(\tau)=\sum_{l=1}^{n}\theta(\tau_l'-\tau)\left[\theta(\tau-\tau_{k_l})+\theta(\tau-\tau_{m_l})\right]$$

Quantum pendulum with constant radius

$$\Xi(\{\tau_k\},\{\tau'_l\}) = \exp\left\{-\int_{-\infty}^{\infty} d\tau \frac{J^{(2n)}(\tau)(N-2) + \left(J^{(2n)}(\tau)\right)^2}{4R_0^2}\right\}$$

Action for correlators of oscillator with quartic anharmonicity

$$S[u;\lambda] = \int_{-\infty}^{\infty} d\tau \Big[-\frac{(\dot{u}(\tau))^2}{4\lambda} - \frac{\left(u^2(\tau) - \omega^2 - \nu(\tau)\right)^2}{4\lambda} + \Big(\frac{N}{2} + J^{(2n)}(\tau) - 1 + \frac{\dot{\nu}(\tau)}{2\lambda}\Big) u(\tau) \Big]$$

supplemented with the external source $\nu(\tau)$ **for O(N) singlets** $(\vec{q})^2(\tau)$

Identities between `` loop'' and local representations of correlators

$$\Big\langle (\vec{q})^2(\tau_1) \Big\rangle_{\vec{q}} = -\frac{\delta}{\delta\nu(\tau_1)} \mathcal{Z}(\eta,\nu)\Big|_{\eta=\nu=0} = \frac{1}{2\lambda}\Big\langle \dot{u}(\tau) - u^2(\tau) + \omega^2 + \nu(\tau) \Big\rangle_u$$

$$= \frac{\delta^2}{\delta^2\vec{\eta}(\tau_1)} \mathcal{Z}(\eta,\nu)\Big|_{\eta=\nu=0} = \Big\langle \frac{N}{2}\int\limits_{\tau_1}^{\infty} d\tau' \exp\left(-2\int\limits_{\tau_1}^{\tau'} d\tau\, u(\tau)\right) \Big\rangle_u$$

Nonlocal ``loop'' integral is identified with a local differential functional !