Generalized Swanson Models and their solutions

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Hermiticity is no longer a necessary condition for energies to be real

$$H \neq H^\dagger$$

Initially, the reality of the spectrum was attributed to the *PT* sym of the Hamiltonian

$$H\,\mathcal{PT} = \mathcal{PT}\,H$$

P : Parity Operator and
T : Time Reversal

$$\mathcal{P}x\mathcal{P} = -x \qquad \mathcal{P}p\mathcal{P} = \mathcal{T}p\mathcal{T} = -p \qquad \mathcal{T}(i.1)\mathcal{T} = -i.1$$

C M Bender & S Boettcher, J Phys A (1998),
C M Bender & S Boettcher, Phys Rev Lett (1998),
Z Ahmed, Phys Lett A (2001), M Znojil (2000),
B Bagchi & C Quesne, Phys Lett A (2000), etc.

- **Unbroken *PT* symmetry : Real Energy**

$$H^*(-x;a) = H(x;a)$$

$$\psi^*(-x;a) = \pm\psi(x;a)$$

- **Spontaneous breakdown of *PT* sym. :**

H is still invariant under *PT,* but not e.fn.

Energies -> **COMPLEX CONJUGATE PAIRS**

Yet, *PT* symmetry is neither NECESSARY nor SUFFICIENT for spectrum to be Real

for non *PT* sym complex pot were found to possess real E, e.g. Complex Morse pot

Q. Is there any necessary & sufficient condition for existence of real energies ?

The required criterion is that *H* should be η pseudo Hermitian

$$H = H^{\sharp} = \eta^{-1} H^{\dagger} \eta$$

$$H^{\dagger} = \eta H \eta^{-1}$$

$$H^{\dagger} \eta = \eta H$$

η is a linear, Hermitian, invertible operator; it is not defined uniquely

A Mostafazadeh, J Math Phys (2002) (2003) etc

- Pseudo Hermiticity of H is equivalent to :

Presence of an Antilinear Symmetry,

PT symmetry being the primary example

L Solombrino, J Math Phys (2002)

- **Conversely**, a quantum system possessing an exact antilinear symmetry, **is pseudo Hermitian**

- Secondly, it is equivalent to a quant system described by a **Hermitian Hamiltonian *h*.**

F G Scholtz, H B Geyer & F J W Hahne, Ann Phys (1992)
R Kretschmer & L Szymanowski, Phys Lett A (2004)
H F Jones, J Phys A (2005)

Let a Sturm-Liouville eq, or a differential op. *H,* act in a complex function space V , endowed with a +ve definite inner product

$$\langle\psi|\eta|\psi\rangle$$

and be described by the Hilbert space $\mathcal{H}$.

Then, there exists a mapping between *H* and *h,* through

$$h = \rho H \rho^{-1}$$

Similarity transformation $\rho = \sqrt{\eta}$

Observables in pseudo Hermitian theory are related to those in the Hermitian theory by

$$\boxed{\mathcal{O} = \rho^{-1}\mathcal{O}_h\ \rho}$$

Recently, Swanson analyzed the non Hermitian, but *PT* sym quadratic Hamiltonian

$$H = \omega a^\dagger a + \alpha a^2 + \beta a^{\dagger\ 2}$$

$$\alpha \neq \beta$$

$a^\dagger$ and a are Harmonic oscillator creation & annihilation operators, and α, β are real const, with dim of inverse time

M S Swanson, J Math Phys (2004)

$$a = \frac{d}{dx} + x$$

$$a^\dagger = -\frac{d}{dx} + x$$

e.v. are real and +ve for

$$\omega^2 \geq 4\alpha\beta$$

Our aim : To study a **pseudo Hermitian generalization** of the Swanson model, with generalized creation and annihilation operators, with real *W(x)*.

$$\mathcal{A} = \frac{d}{dx} + W(x)$$

$$\mathcal{A}^{\dagger} = -\frac{d}{dx} + W(x)$$

W(x) is the pseudo superpotential

$$\boxed{W(x) = -\frac{{f_0}'(x)}{f_0(x)}}$$

$\boxed{f_0(x)}$ is the G.S. wave fn of

$$\boxed{\mathbf{H} = \mathcal{A}^\dagger \mathcal{A}}$$

- Give a general formalism for solving such an equation, by mapping it to a Hermitian system with known solutions

- Examine the range of values of the parameters α, β for which the energies are real

- Find the similarity transformation mapping *H* and *h*

- Find an operator η with respect to which *H* is pseudo Hermitian

Since real E occur for certain parameter values only, hence similar to CES models

Our starting eqn

$$H = \mathcal{A}^{\dagger}\mathcal{A} + \alpha\mathcal{A}^{2} + \beta\mathcal{A}^{\dagger\ 2}$$

$$\alpha \neq \beta$$

$$H\psi \quad = \quad \left\{ - (1-\alpha-\beta)\left(\frac{d}{dx} - \frac{\alpha-\beta}{1-\alpha-\beta}\, W\right)^{2} \right.$$

$$\left. + \frac{1-4\alpha\beta}{1-\alpha-\beta}\, W^{2} - \; W' \right\}\psi$$

$$= \quad E\psi$$

We perform the gauge transformation

$$\psi(x) = e^{\mu \int W(x)dx} \phi(x)$$

$$\mu = \frac{\alpha - \beta}{1 - \alpha - \beta}$$

to reduce it to the well known Schro. form

$$h \ \phi(x) \ = \ \left(-\frac{d^2}{dx^2} \ + \ V(x)\right) \phi(x) \ = \ \varepsilon\phi(x)$$

$$V(x) = \left(\frac{\sqrt{1-4\alpha\beta}}{1-\alpha-\beta} W(x)\right)^2 - \frac{1}{1-\alpha-\beta} W'(x)$$

$$\varepsilon = \frac{E}{1-\alpha-\beta}$$

From SUSY QM, *h* can always be written as

$$\begin{aligned} h &= A^\dagger A + \epsilon \\ &= -\frac{d^2}{dx^2} + \mathrm{w}^2 - \mathrm{w}' + \epsilon \end{aligned}$$

$$A = \frac{d}{dx} + \mathrm{w}(x)$$

$$A^{\dagger} = -\frac{d}{dx} + \mathrm{w}(x)$$

$$\mathrm{w}(x) = -\frac{d \ln \varphi_0(x)}{dx}$$

φ_0 is the GS e.fn of $A^{\dagger}A$ with energy ε_0.

Task : To identify V(x) with an exactly solvable potential, & thus find the solutions of *h*

For further convenience, identify V(x) with SIP, as using SUSY QM, the raising and lowering operator method of HO can be generalized to a whole class of SIP, which includes all the analytically solvable models.

- To narrow down the class of potentials further, our strategy is to write V(x) in SUSY form

$$\boxed{\mathrm{w}^2(x) - \mathrm{w}'(x)}$$

- This imposes certain restrictions on the permissible values of the parameters, irrespective of the explicit form of W(x)

$$\boxed{\alpha + \beta \ < \ 1} \qquad \boxed{4\alpha\beta \ < \ 1}$$

- Some additional constraints might arise due to normalization requirement of the wave fn, depending on the particular model

Check :

Model based on **Harmonic Oscillator**

$$V(x) = x^2 - 1$$

with

$$W(x) = c_1 x$$

Results :

$$V(x) = \left(\frac{\sqrt{1 - 4\alpha\beta}}{1 - \alpha - \beta} c_1 x \right)^2 - \frac{c_1}{1 - \alpha - \beta}$$

$$c_1 = \frac{1 - \alpha - \beta}{\sqrt{1 - 4\alpha\beta}}$$

$$\psi(x) = e^{\frac{c_1 \mu}{2} x^2} \phi(x) = e^{\frac{1}{2} \frac{\alpha - \beta}{\sqrt{1-4\alpha\beta}} x^2} \phi(x)$$

$$\epsilon_n = 2n + 1 - \frac{1}{\sqrt{1 - 4\alpha\beta}}$$

For wave fn to be normalizable $\alpha < \beta$

Energies are real only if $4\alpha\beta < 1$

Explicit example :

Model based on **trigonometric Rosen Morse I**

$$V(x) \;=\; A\,(A-1)\;\csc^2 x \;+\; 2B\;\cot x - A^2 + \frac{B^2}{A^2}$$

$$A > 0\;,\;\; B > 0$$

$$0 \le x \le \pi$$

For constructing the creation & annihilation operators

$$W(x) = -A_1 \cot x - \frac{B_1}{A_1}$$

$$A_1 > 0 \ , \ B_1 > 0$$

Substitution yields

$$\psi(x) = e^{-\mu_1 x} \sin^{\mu_2} x \quad \phi(x)$$

$$\mu_1 = \frac{B_1}{A_1} \frac{(\alpha - \beta)}{(1 - \alpha - \beta)}$$

$$\mu_2 = -\frac{A_1(\alpha - \beta)}{(1 - \alpha - \beta)}$$

For wave fn to be well defined at the boundaries

$\mu_2 > 0$ so that $\alpha < \beta$

$$V(x) = \frac{A_1^2 (1 - 4\alpha\beta) - A_1 (1 - \alpha - \beta)}{(1 - \alpha - \beta)^2} \csc^2 x$$

$$+ \; 2B_1 \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \cot x$$

$$- \left(A_1^2 - \frac{B_1^2}{A_1^2} \right) \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2}$$

Provided one makes the identification

$$A = \frac{1}{2} \pm \frac{\sqrt{1+4\sigma}}{2}$$

$$B = B_1 \frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$$

$$\sigma = \frac{A_1^2 (1-4\alpha\beta) - A_1 (1-\alpha-\beta)}{(1-\alpha-\beta)^2}$$

Since $A > 0$,

+ve sign only is allowed in A

For existence of bound states, $\sigma > 0$

Furthermore, since $A_1 \neq 0$

hence $$A_1 > \frac{1 - \alpha - \beta}{1 - 4\alpha\beta}$$

Solutions of *h*

$$\varepsilon_n = (A+n)^2 - \frac{B^2}{(A+n)^2} - \left(A_1^2 - \frac{B_1^2}{A_1^2}\right)\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$$

$$\phi_n(x) \approx \left(y^2-1\right)^{-\frac{(A+n)}{2}} e^{\left(\frac{B}{A+n}\right)x} P_n^{(s_+\ ,\ s_-)}(y)$$

$$y = i\ \cot x$$

$$s_\pm = -A - n \pm i\frac{B}{(A+n)}$$

Thus *H* has solutions

$$E_n = (1 - \alpha - \beta)\,\varepsilon_n$$

$$\psi_n(x) \;\approx\; e^{\left\{\frac{B}{(A+n)} - \mu_1\right\}x} \sin^{A+n+\mu_2} x \; P_n^{(s_+\;,\;s_-)}(y)$$

Choice of parameters :

α	β	$\alpha+\beta$	$4\alpha\beta$	A_1	B_1	μ_1	μ_2	σ	A	B	E_n
1/4	1/2	3/4	1/2	3/2	1/8	- 1/12	3/2	12	4	1	$\frac{1}{4}\varepsilon_n$
1/4	2/3	11/12	2/3	1	1/2	-5/2	5	36	6.52	24	$\frac{1}{12}\varepsilon_n$
1/8	3/4	7/8	3/8	1	2	-10	5	32	6.18	80	$\frac{1}{8}\varepsilon_n$
1/3	1/2	5/6	2/3	1	2	-2	1	6	3	36	$\frac{1}{6}\varepsilon_n$

In each case, the parameters satisfy the constraints

$$\alpha+\beta < 1 \qquad 4\alpha\beta < 1 \qquad \alpha < \beta$$

$$A_1 > \frac{1-\alpha-\beta}{1-4\alpha\beta}$$

Similarity transformation between *H* and *h*

Let us focus our attention on the

transformation ρ relating $\psi(x)$

and $\phi(x)$

$$\phi(x) = \rho\ \psi(x)$$

$$\rho = e^{-\mu \int W dx}$$

$$\mu = \frac{\alpha - \beta}{1 - \alpha - \beta}$$

If

$$H\psi = E\psi$$

then

$$h\phi = E\phi$$

provided

$$h = \rho\, H\, \rho^{-1}$$

For example, for

$$\alpha = \frac{1}{4}\ ,\ \beta = \frac{1}{2}\ ,\ A_1 = \frac{3}{2}\ ,\ B_1 = \frac{1}{8},$$

$$H\psi(x) = E\psi(x)$$

$$= \left\{ -\frac{1}{4}\frac{d^2}{dx^2} + \left(\frac{18\ \cot\ x + 1}{24}\right)\frac{d}{dx}\right.$$

$$\left. + \frac{33}{16}(\csc x)^2 + \frac{7}{16}\cot x - \frac{2261}{576}\right\}\psi(x)$$

$$h\phi(x) = \left\{-\frac{d^2}{dx^2} + 12\ \csc^2 x\ +\ 2\ \cot x - \frac{323}{18}\right\}\phi(x) = \varepsilon\phi(x)$$

The two solutions are related by

$$\psi(x) = e^{\frac{1}{12}x}\sin^{\frac{3}{2}x}\phi(x)$$

and

$$E = \frac{1}{4}\varepsilon$$

Q. Is *H* actually pseudo Hermitian ?

Let us explore the relationship between

$$H \;=\; \mathcal{A}^\dagger \mathcal{A} \;+\; \alpha \mathcal{A}^2 \;+\; \beta \mathcal{A}^{\dagger\,2}$$

and its adjoint

$$H^\dagger = \mathcal{A}\mathcal{A}^\dagger + \alpha \mathcal{A}^2 + \beta \mathcal{A}^{\dagger\,2}$$

If we put

$$\eta = \rho^2 = e^{-2\mu \int W dx}$$

then $$H^\dagger \, \eta \; = \; \eta \, H$$

i.e. $$H^\dagger \; = \; \eta \, H \, \eta^{-1}$$

Special Case : *PT* invariant Model

$\mathcal{A}$ and $\mathcal{A}^\dagger$ Transform under parity and time reversal as

$$\boxed{\mathcal{P} \;:\; \mathcal{A}\,(\mathcal{A}^\dagger) \;\rightarrow\; -\,\mathcal{A}\,(\mathcal{A}^\dagger)}$$

$$\boxed{\mathcal{T} \;:\; \mathcal{A}\,(\mathcal{A}^\dagger) \;\rightarrow\; \mathcal{A}\,(\mathcal{A}^\dagger)}$$

For *H* to be invariant under *PT*

W(x) must transform as

$$(\mathcal{PT})\ W(x)\ (\mathcal{PT})^{-1} = -\ W(x)$$

Hence, for the model considered here,

$$B_1 = 0$$

$$W(x) = -A_1 \cot x \qquad A_1 > 0$$

The potential assumes the simple form

$$\boxed{V(x) \;=\; A(A+1)\;\csc^2 x - A^2}$$

with $$\boxed{\varepsilon_n = (A+n)^2 - A^2}$$

Thus, the solutions of *H* are obtained as

$$\psi_n(x) \approx (\sin x)^{A+n+\mu_2} \; P_n^{(-A-n,-A-n)}(i \; \cot x)$$

with energies

$$E_n = \frac{1}{1-\alpha-\beta} \; \varepsilon_n$$

Conclusions :

- Solved a pseudo Hermitian eqn exactly, by mapping it to an equivalent Hermitian one

- Found the similarity transformation ρ mapping the Non Hermitian H to its Hermitian equivalent h

- **Found a simple way to calculate an exact form of** η

- However, the parameters must obey certain conditions.

Further Extensions :

- To find isospectral partner of the non Hermitian Hamiltonian $H(x)$, starting from the isospectral partner of $h(x)$

- To generalize this formalism further

by taking complex pseudo superpotential

$$H = \mathcal{B}\mathcal{A} + \alpha\mathcal{A}^2 + \beta\mathcal{B}^2$$

with complex α, β

$$\mathcal{B} = -\frac{d}{dx} + W(x)$$

$$\mathcal{A} = \frac{d}{dx} + W(x)$$

For real energies

$$\alpha = \beta^*$$

$$\mathcal{B} = \mathcal{A}^\sharp = \eta^{-1}\mathcal{A}^\dagger\eta$$

T H A N K S