Nonlocal Variant of $\mathcal{PT}$ Symmetric Potential

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The time independent Schrödinger equation in the position representation is given by (assuming $\hbar = 2m = 1$)

$$\tilde{H}\psi(x) = E\psi(x) \tag{1}$$

where the Hamiltonian $\tilde{H}$ is given by [Choi et al, Phys.Rev.A60 (1999) 796]

$$\tilde{H}\psi(x) = -\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) + \int_{-\infty}^{\infty} dy v(x,y)\psi(y) = E\psi(x) \tag{2}$$

$V(x)$ and $v(x,y)$ being complex local and nonlocal potentials respectively.

We now consider a pair of Hamiltonians $\tilde{H}_{\pm}$ of the form (1) where $\tilde{H_+} = AB$, $\tilde{H_-} = BA$, A and B being linear first order differential operators. Then $\tilde{H_-}$ is the isospectral partner of $\tilde{H_+}$.

Correspondingly, the Schrödinger equations for the partner Hamiltonians $\tilde{H}_+$ and $\tilde{H}_-$ are given by

$$\begin{aligned}\tilde{H}_+\psi_+(x) &= -\frac{d^2\psi_+(x)}{dx^2} + V_+(x)\psi_+(x) + \int_{-\infty}^{\infty} dy v_+(x,y)\psi_+(y) \\ &= E_+\psi_+(x)\end{aligned} \tag{3}$$

and

$$\begin{aligned}\tilde{H}_-\psi_-(x) &= -\frac{d^2\psi_-(x)}{dx^2} + V_-(x)\psi_-(x) + \int_{-\infty}^{\infty} dy v_-(x,y)\psi_-(y) \\ &= E_-\psi_-(x)\end{aligned} \tag{4}$$

Writing [Choi et al, Phys.Rev.A60 (1999) 796]

$$
\begin{aligned}
< x|V_\pm(x)|\psi_\pm > &= V_\pm(x)\psi_\pm(x) \\
< x|v_\pm(x,y)|\psi_\pm > &= \int_{-\infty}^{\infty} dy v_\pm(x,y)\psi_\pm(y)
\end{aligned} \tag{5}
$$

equations (3) and (4) can be cast into the form

$$
\frac{d^2\psi_\pm(x)}{dx^2} + < x|V_\pm(x)|\psi_\pm > + < x|v_\pm(x,y)|\psi_\pm > = E_\pm\psi_\pm(x) \tag{6}
$$

Let us write the potentials in operator form as

$$
\begin{aligned}
\hat{V}_\pm &= \int_{-\infty}^{\infty} dx|x > V_\pm(x) < x| \\
\hat{v}_\pm &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy|x > v_\pm(x,y) < y|
\end{aligned} \tag{7}
$$

Then the partner Hamiltonians $\tilde{H}_+$ and $\tilde{H}_-$ can be factorized respectively as

$$\tilde{H}_+ = \hat{A}\hat{B} = \hat{p}^2 + \hat{V}_+ + \hat{v}_+ \tag{8}$$

and

$$\tilde{H}_- = \hat{B}\hat{A} = \hat{p}^2 + \hat{V}_- + \hat{v}_- \tag{9}$$

where the first order differential operators $\hat{B}$ and $\hat{A}$ are defined by

$$\begin{aligned} \hat{B} &= -i\hat{p} + \hat{W} + \hat{w} \\ \hat{A} &= i\hat{p} + \hat{W} + \hat{w} \end{aligned} \tag{10}$$

And

$$\begin{aligned}\hat{W} &= \int_{-\infty}^{\infty} dx|x> W(x) < x| \\ \hat{w} &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy|x> w(x,y) < y|\end{aligned} \tag{11}$$

The potentials $V_\pm$ and $v_\pm$ are written in terms of the factorization potentials $W(x)$ and $w(x,y)$ as

$$\begin{aligned}V_\pm(x) &= [W(x)]^2 \pm \frac{dW(x)}{dx} \\ v_\pm(x,y) &= \int_{-\infty}^{\infty} du w(x,u)w(u,y) + [W(x)+W(y)]w(x,y) \\ &\pm \left[\frac{\partial w(x,y)}{\partial x} + \frac{\partial w(x,y)}{\partial y}\right]\end{aligned} \tag{12}$$

It is to be mentioned here that in models with local potentials, there is a one-to-one relationship between the ground state and the factorization potential $W(x)$ but it is not so with non local potentials. In non local models, given the factorization potentials $W(x)$ and $w(x,y)$, the zero-energy ground state $\psi_0(x)$ is obtained from the integro-differential equation

$$\frac{d\psi_0(x)}{dx} + W(x)\psi_0(x) + \int_{-\infty}^{\infty} dy w(x,y)\psi_0(y) = 0 \tag{13}$$

We shall now construct a class of exactly solvable models with both complex local and nonlocal potentials starting from any exactly solvable local model with factorization potential $W_0(x)$ and to this end we choose

$$\begin{aligned} W(x) &= (1-c)W_0(x) \\ w(x,y) &= C_1 \frac{\partial}{\partial x}\delta(x-y) \end{aligned} \tag{14}$$

where C_1 is a parameter of nonlocality and c is a constant.

Substituting $W(x)$ and $w(x,y)$ we obtain

$$
\begin{aligned}
V_{\pm}(x) &= (1-c)^2[W_0(x)]^2 \pm (1-c)\frac{dW_0(x)}{dx} \\
\int_{-\infty}^{\infty} dy v_{\pm}(x,y)\psi_{\pm}(y) &= C_1^2\frac{d^2\psi_{\pm}(x)}{dx^2} + 2C_1(1-c)W_0(x)\frac{d\psi_{\pm}(x)}{dx} \\
&\pm(1-c)C_1\frac{dW_0(x)}{dx}\psi_{\pm}(x)
\end{aligned}
\tag{15}
$$

Consequently the eigenvalue equations for the Hamiltonians $\tilde{H}_\pm$ are written as

$$\tilde{H}_\pm \psi_\pm = E_\pm \psi_\pm \tag{16}$$

where

$$\begin{aligned} \tilde{H}_\pm &= -(1-C_1^2)\frac{d^2}{dx^2} + 2C_1(1-c)W_0(x)\frac{d}{dx} \\ &+ (1-c)(C_1 \pm 1)\frac{dW_0(x)}{dx} + (1-c)^2 W_0^2(x) \end{aligned} \tag{17}$$

To solve this eigenvalue problem our strategy would be to find a similarity transformation mapping the Hamiltonians $\tilde{H}_\pm$ into standard form.

To this end we make the transformation

$$\psi_\pm(x) = e^{-\frac{1}{2}\int f(x)dx}\phi_\pm(x) = \eta\phi_\pm(x) \tag{18}$$

where

$$\begin{aligned} \eta &= e^{-\frac{1}{2}\int f(x)dx} \\ f(x) &= -\frac{2C_1(1-c)}{(1-C_1^2)} W_0(x) \end{aligned} \tag{19}$$

so that the transformed Hamiltonians $\bar{H}_\pm = \eta^{-1} \tilde{H}_\pm \eta$ are given by

$$\begin{aligned} \bar{H}_\pm \phi_\pm(x) &= -(1-C_1^2)\frac{d^2\phi_\pm(x)}{dx^2} + \frac{(1-c)^2}{(1-C_1^2)} W_0^2(x)\phi_\pm(x) \\ &\pm(1-c)W_0^{'}(x)\phi_\pm(x) = E_\pm \phi_\pm + (x) \end{aligned} \tag{20}$$

It is to be noted that $\bar{H}_\pm$ can be factorised as

$$\bar{H}_+ = CD \qquad \bar{H}_- = DC \tag{21}$$

where

$$\begin{aligned} C &= (1+C_1)\frac{d}{dx} + \frac{(1-c)}{(1-C_1)}W_0(x) \\ D &= (1-C_1)\frac{d}{dx} + \frac{(1-c)}{(1+C_1)}W_0(x) \end{aligned} \tag{22}$$

For the special case $C_1^2 = c$, the Hamiltonians $\bar{H}_\pm$ can be written in terms of the local Hamiltonians $\bar{H}_{\pm,\mathrm{local}}$ as

$$\bar{H}_\pm = (1-c)\bar{H}_{\pm,\mathrm{local}} \tag{23}$$

Then it can be shown that

$$\begin{aligned} \phi_\pm &= \chi_\pm \\ E_\pm &= (1-c)E_{\pm,\mathrm{local}} \end{aligned} \tag{24}$$

Now to find eigenvalues $E_\pm$ and eigenfunctions $\psi_\pm$ of the Schroedinger equation (16), we proceed as follows: For local Hamiltonians we already have

$$\bar{H}_{\pm,\text{local}} \;\; \chi_{\pm,\text{local}} \; = E_{\pm,\text{local}} \;\; \chi_{\pm,\text{local}} \tag{25}$$

where $\chi_{\pm,\text{ local}}$ are the eigenfunctions of the local Hamiltonians. Therefore

$$\bar{H}_\pm \;\; \chi_{\pm,\text{local}} \; = (1-c)\bar{H}_{\pm,\text{local}} \;\; \chi_{\pm,\text{local}} \; = (1-c)E_{\pm,\text{local}} \;\; \chi_{\pm,\text{local}} \tag{26}$$

Putting (18) into (16) we have

$$\begin{aligned} \tilde{H}_\pm \eta \phi_\pm &= E_\pm \eta \phi_\pm \\ &= \eta E_\pm \phi_\pm \end{aligned} \tag{27}$$

Operating η^{-1} from the left of the above equation, we get

$$\begin{aligned} \bar{H}_{\pm}\phi_{\pm} &= E_{\pm}\phi_{\pm} \\ &= (1-c)\bar{H}_{\pm,\mathrm{local}}\ \phi_{\pm} \end{aligned} \tag{28}$$

Comparing (28) with (26) we have

$$\begin{aligned} \phi_{\pm} &= \chi_{\pm} \\ E_{\pm} &= (1-c)E_{\pm,\mathrm{local}} \end{aligned} \tag{29}$$

Example 1.

Here we shall apply the above formalism to obtain eigenvalues and eigenfunctions of a nonlocal variant of the $\mathcal{PT}$ symmetric Rosen Morse potential [Znojil, J.Phys. A33, (2000) L61]

In this case the factoriozation potential is taken as

$$W_0(x) = i\frac{B}{A} + Atanhx \qquad (30)$$

Then $\tilde{H}_\pm$ are given by

$$\tilde{H}_\pm = -(1-C_1^2)\frac{d^2\phi_\pm(x)}{dx^2} + C_1(1-c)(\frac{iB}{A} + Atanhx)\frac{d}{dx}$$
$$+(1-c)(C_1 \pm 1)Asech^2x + (1-c)^2A^2sech^4x \qquad (31)$$

With $f(x) = -\frac{2C_1(1-c)}{(1-C_1^2)}(i\frac{B}{A} + Atanhx)$ the transformed Hamiltonians are given by

$$\bar{H}_\pm = -(1-C_1^2)\frac{d^2}{dx^2} + \frac{(1-c)^2}{(1-C_1^2)^2}(\frac{iB}{A} + tanhx)^2 \pm (1-c)Asech^2x \tag{32}$$

These two Hamiltonians admit the factorisations

$$\bar{H}_\pm = \left[(1+C_1)\frac{d}{dx} + \frac{(1-c)}{(1\mp C_1)}(\frac{iB}{A} + Atanhx)\right] \tag{33}$$

$$\left[\mp(1\mp C_1)\frac{d}{dx} + \frac{(1-c)}{(1\pm C_1)}(\frac{iB}{A} + Atanhx))\right]$$

With $C_1^2 = c$, the eigenvalue equations for $\bar{H}_\pm$ can be written as

$$\bar{H}_\pm \phi_\pm(x) = (1-c)\Big[-\frac{d^2\phi_\pm(x)}{dx^2} + (\frac{iB}{A} + Atanhx)^2\phi_\pm(x) \\ \pm(A\ sech^2 x)\phi_\pm(x)\Big] = E_\pm \phi_\pm(x) \tag{34}$$

The eigenvalues and eigenfunctions can be written using the results of corresponding local model [Levai, J.Phys. A22, (1989) 689, Znojil, J.Phys. A33, (2000) L61] For unbroken $\mathcal{PT}$ symmetry

$$\psi_-(x) = \frac{1}{(coshx)^{A(1-\sqrt{c})} - n} e^{iBx(\frac{\sqrt{c}}{A} - \frac{1}{A-n})} \tag{35}$$

$$P_n^{A-n+\frac{iB}{A-n}, A-n-\frac{iB}{A-n}}(tanhx)$$

where $P_n^{\alpha,\beta}$ is the Jacobi polynomial.

$$E_- = (1-c)[A^2 - \frac{B^2}{A^2} - (A-n)^2 + \frac{B^2}{(A-n)^2}] \tag{36}$$

where $n = 0, 1, 2 \cdots < A(1-\sqrt{c})$

And

$$\psi_+(x) = \frac{1}{(coshx)^{(A-1)(1-\sqrt{c})} - n} e^{iBx(\frac{\sqrt{c}}{A-1} - \frac{1}{A-1-n})} \tag{37}$$

$$P_n^{A-1-n+\frac{iB}{A-1-n}, A-1-n-\frac{iB}{A-1-n}}(tanhx)$$

$$E_+ = (1-c)[(A^2 - \frac{B^2}{A^2} - (A-1-n)^2 + \frac{B^2}{(A-1-n)^2}] \tag{38}$$

where $n = 0, 1, 2 \cdots < (A-1)(1-\sqrt{c})$

Example 2. Scarf Potential

In this case the factorisation potential is taken as

$$W_0(x) = \lambda tanhx + i\mu sechx \tag{39}$$

Then $\tilde{H}_\pm$ are given by

$$\tilde{H}_\pm = -(1 - C_1^2)\frac{d^2}{dx^2} + 2C_1(1-c)(\lambda tanhx + i\mu sechx)\frac{d}{dx} \tag{40}$$
$$+(1-c)(C_1 \pm 1)(\lambda sech^2x - i\mu sechxtanhx)$$

$$+(1-c)^2(\lambda tanhx + i\mu sechx)^2$$

With $f(x) = -\dfrac{2C_1(1-c)}{(1-C_1^2)}(\lambda tanhx + i\mu sechx)$ the transformed Hamiltonians are given by

$$\bar{H}_{\pm} = -(1-C_1^2)\frac{d^2}{dx^2} + \frac{(1-c)^2}{(1-C_1^2)^2}(\lambda tanhx + i\mu sechx)^2 \pm(1-c)(\lambda sech^2x - i\mu sechxtanhx) \tag{41}$$

These two Hamiltonians admit the factorisations

$$\tilde{H}_{\pm} = \left[\pm(1 \pm C_1)\frac{d}{dx} + \frac{(1-c)}{(1 \mp C_1)}(\lambda\ tanh\ x + i\mu\ sech\ x)\right] \tag{42}$$

$$[\mp(1 \mp C_1)\frac{d}{dx} + \frac{(1-c)}{(1 \pm C_1)}(\lambda\ tanh\ x + i\mu\ sech\ x)]$$

With $C_1^2 = c$, the eigenvalue equations for $\bar{H}_\pm$ can be written as

$$\bar{H}_\pm \phi_\pm(x) = (1-c)\left[-\frac{d^2\phi_\pm(x)}{dx^2} + (\lambda tanhx + i\mu sechx)^2\phi_\pm(x)\right.$$
$$\left.\pm(\lambda sech^2 x - i\mu sechxtanhx)\phi_\pm(x)\right] = E_\pm\phi_\pm(x) \tag{43}$$

The eigenvalues and eigenfunctions can be written using the results of the corresponding local model [Z.Ahmed, Phys.Letts. A282 (2001) 343, A287 (2001) 295].

If the potential in the above equation is written as $V(x) = V_1 sech^2 x + iV_2 sechxtanhx$, then for unbroken $\mathcal{PT}$ symmetry $|V_2| \leq V_1 + \frac{1}{4}$, V_1,V_2 being given in the following equations. Three cases will arise

Case 1. Positive square roots taken in both t and s (given in the following equations). In this case the eigenvalues are given by

$$E_{\pm,n^+} = (1-c)[\lambda^2 - \{n^+ + \frac{1}{2} - \frac{1}{2}(t+s)\}^2] \tag{44}$$

where

$$\begin{aligned} n^+ &= 0,1,2\cdots < \tfrac{s+t-1}{2} \\ V_1 &= (\lambda^2+\mu^2) \mp \lambda \\ V_2 &= \pm\mu - 2\lambda\mu \\ t &= \sqrt{\tfrac{1}{4}+V_1-V_2} \\ s &= \sqrt{\tfrac{1}{4}+V_1+V_2} \end{aligned} \tag{45}$$

and the eigenfunctions corresponding to these real eigenvalues are

$$\psi_{\pm,n^+}(x) = (sech\ x)^{\frac{(1-\sqrt{c})(s+t)-(1\pm\sqrt{c})}{2}}$$

$$\exp\left[\frac{i}{2}(1+\sqrt{c})(t-s)tan^{-1}(sinhx)\right] P_n^{(-t,-s)}(isinhx) \tag{46}$$

where $P_n^{(-t,-s)}$ denotes Jacobi polynomial.

Case 2. $V_2 > 0$, positive square root in s and negative square root in t.
In this case

$$E_{\pm,n^-} = (1-c)[\lambda^2 - \{n^- + \frac{1}{2} - \frac{1}{2}(s-t)\}^2] \tag{47}$$

where

$$\begin{aligned} n^- &= 0, 1, 2 \cdots < \tfrac{s-t-1}{2} \\ V_1 &= (\lambda^2 + \mu^2) \mp \lambda \\ V_2 &= \pm\mu - 2\lambda\mu \\ t &= -\sqrt{\tfrac{1}{4} + V_1 - V_2} \\ s &= \sqrt{\tfrac{1}{4} + V_1 + V_2} \end{aligned} \tag{48}$$

and

$$\psi_{\pm,n^-}(x) = (sech\ x)^{\frac{(1-\sqrt{c})(s-t)-(1\pm\sqrt{c})}{2}}$$

$$\exp\left[-\frac{i}{2}(1-\sqrt{c})(t+s)tan^{-1}(sinhx)\right]P_n^{(t,-s)}(isinhx) \quad (49)$$

Case 3. $V_2 < 0$, positive square root taken in t and negative square root in s.
In this case

$$E_{+,n^-} = (1-c)[\lambda^2 - \{n^- + \frac{1}{2} - \frac{1}{2}(t-s)\}^2] \tag{50}$$

where

$$\begin{aligned} n^- &= 0,1,2\cdots < \tfrac{t-s-1}{2} \\ V_1 &= (\lambda^2+\mu^2) \mp \lambda \\ V_2 &= \pm\mu - 2\lambda\mu \\ t &= \sqrt{\tfrac{1}{4} + V_1 - V_2} \\ s &= -\sqrt{\tfrac{1}{4} + V_1 + V_2} \end{aligned} \tag{51}$$

and

$$\psi_{\pm,n^-}(x) = (sech\ x)^{\frac{(1-\sqrt{c})(t-s)-(1\pm\sqrt{c})}{2}}$$

$$\exp\left[\frac{i}{2}(1+\sqrt{c})(t+s)tan^{-1}(sinhx)\right]P_n^{(-t,s)}(isinhx) \quad (52)$$

Conclusion

1.The complex factorization approach is formally extended to complex nonlocal Hamiltonians. Many theoretical questions for nonlocal potentials are yet to be resolved e.g. inner product for $\mathcal{PT}$ symmetric nonlocal systems, formulation of nonlocal $\mathcal{PT}$ symmetric supersymmetry and pseudo-supersymmetry

2. However the present framework provides a link, albeit for special value of the nonlocal parameter, the nonlocal potential to a corresponding local $\mathcal{PT}$ symmetric potential.