Complexification of Energies for Relativistic Hamiltonians

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The matrix case

- Strongly preferred is the Jordan-block complexification. The reason in 2x2 case is that the diagonalisability of the matrix Hamiltonian of the type $H = H_0 + cV$ at the exceptional point would imply the commutativity of H_0 and V. This is clearly not satisfied in most realistic situations. In more than 2 dimensions the situation is more complicated, however the "diagonalisable" complexification is still prohibited. It is the same reason as the prohibition of level crossings for Hermitean operators.
- Energy dependence near the exceptional point is usually well approximated by the square-root function.
- Example: 2x2 matrix pseudo-Hermitean with respect to σ_3:

$$\mathsf{H} = \begin{pmatrix} a & c e^{iq} \\ -c e^{-iq} & b \end{pmatrix}$$

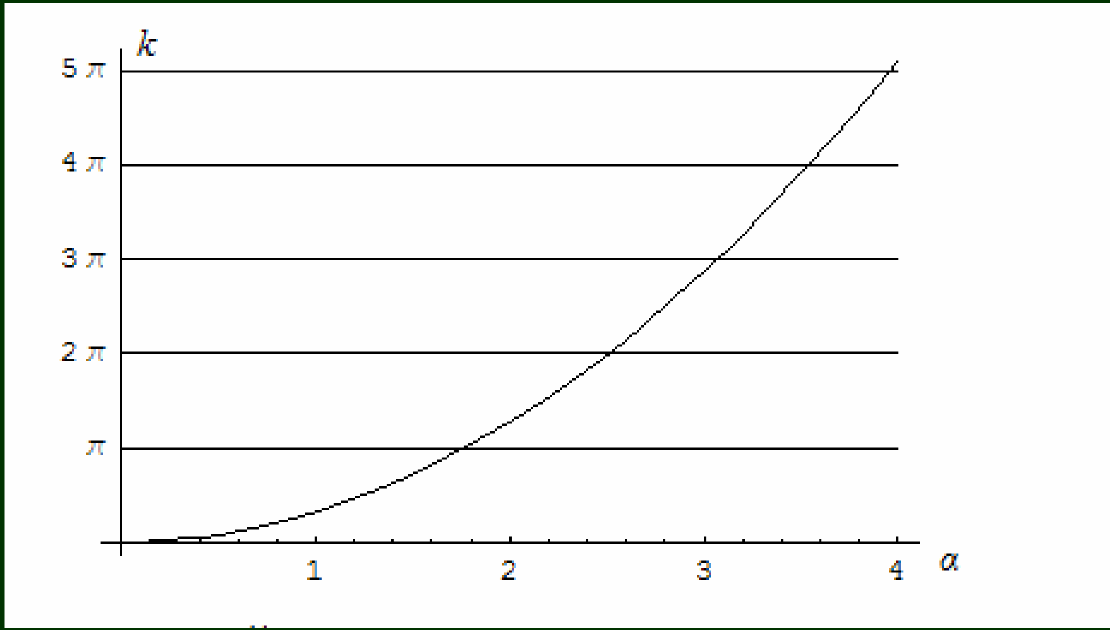
$$E = \begin{cases} \frac{a+b\pm S}{2} & \text{for } T > 2c \\ \frac{a+b\pm i S}{2} & \text{for } T < 2c \end{cases}$$

$$T = |a-b| \qquad S = |\sqrt{T^2 - 4c^2}|$$

Schrödinger case

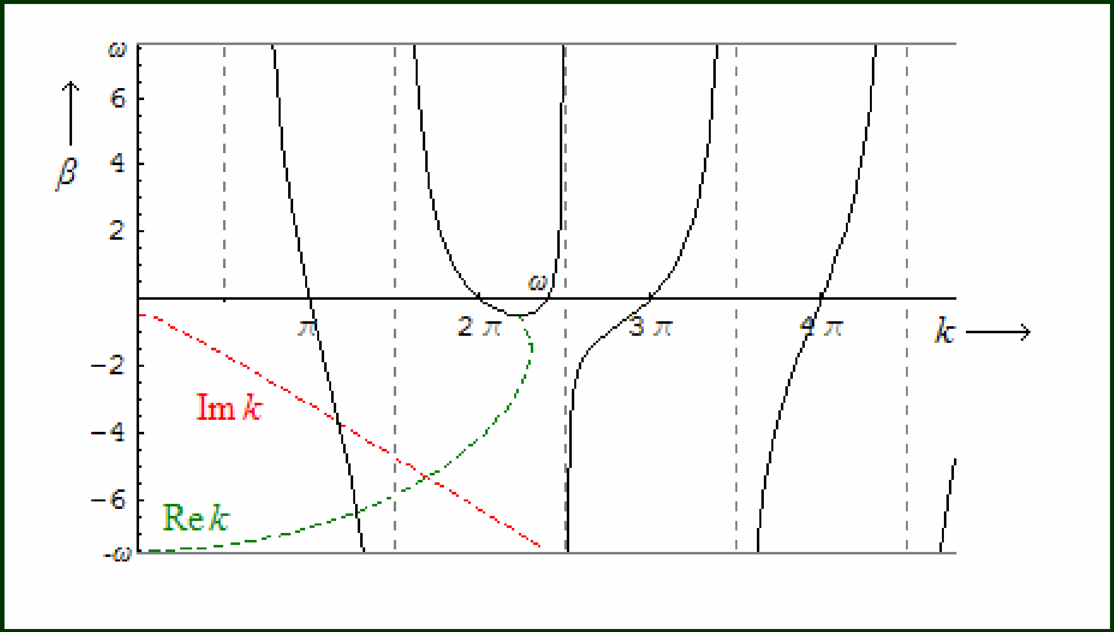
- An exceptional point without complexification: parametrically dependent boundary conditions (Krejčiřík et al.):

$$\psi'(0) + (\beta + i\alpha)\psi(0) = 0$$
$$-\psi'(d) + (\beta - i\alpha)\psi(d) = 0$$



- For β different from 0 a standard complexification occurs.

- The nonusual behaviour is singular.



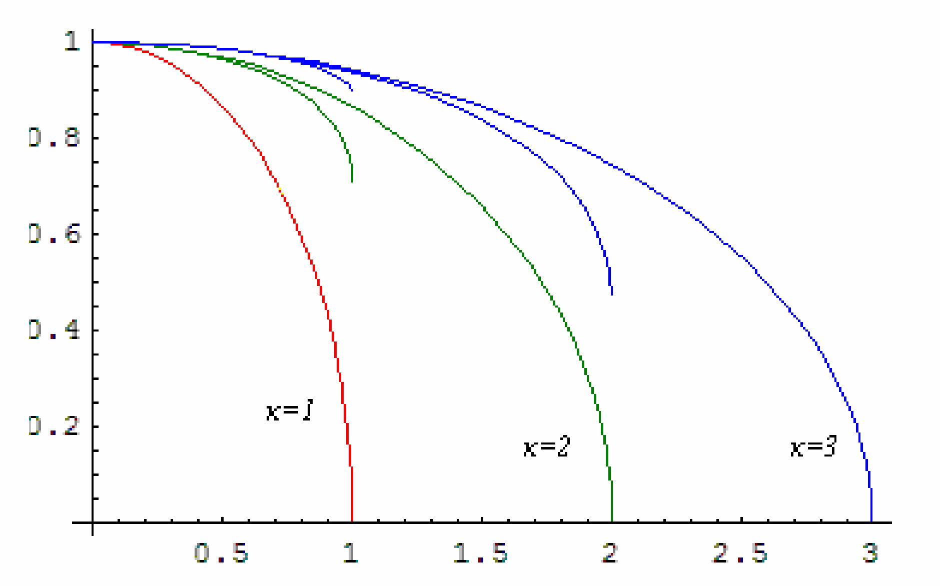
Relativistic systems – a motivation to study

- The radial Coulomb Hamiltonian is

$$\mathsf{H} = \begin{pmatrix} -\frac{Z}{r} - M & -\partial_r - \frac{\kappa}{r} \\ \partial_r - \frac{\kappa}{r} & -\frac{Z}{r} + M \end{pmatrix}$$

$$\mathrm{Dom}\,\mathsf{H} \in \{\psi \in L_2(\mathbb{R}^+) \oplus L_2(\mathbb{R}^+) | \psi' \in L_2(\mathbb{R}^+) \oplus L_2(\mathbb{R}^+)\}$$

- The spectrum for $\kappa = 1, 2, 3$ (ie. *s*, *p*, *d*-states) looks like this (color distinguishes the principal quantum number)



- Complexification occurs for all eigenvalues with a given κ simultaneously.
- The structure of the spectrum near the exceptional point is square-root-like.

$$\gamma = \sqrt{\kappa^2 - Z^2}$$

$$E_n = \left(1 + \frac{Z}{n - \kappa + \gamma}\right)^{-1/2}$$

- Why is H not self-adjoint? The problem is in the origin ($r = 0$). Square integrability of $H\psi$ forces $\psi(0) = 0$ for ψ in Dom H, but only if $\gamma^2 > 1/2$.

- There is only one real eigenvalue (but two complex) near the EP!

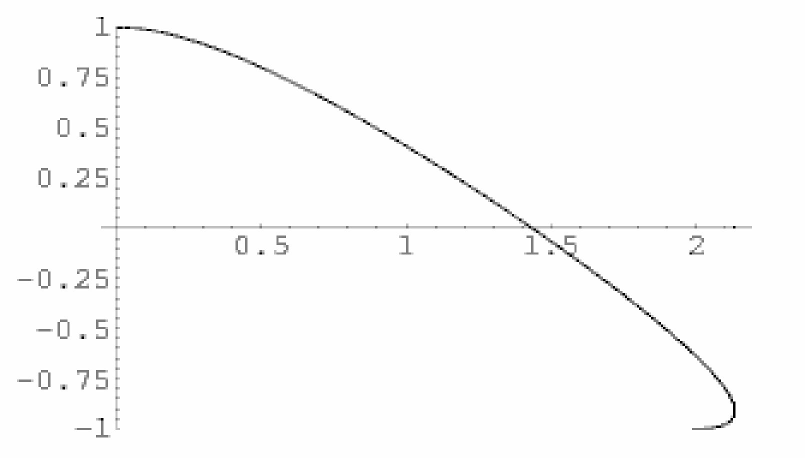
Square well - The Klein & Gordon case

- One dimensional Klein-Gordon equation

$$((E + C\chi_{(0,d)}(x))^2 - \partial_x^2 - M^2)\psi(x) = 0$$

- We put M = 1, and the real secular equation we get is,

$$\arctan\sqrt{\frac{(1+E)(1-E)}{(C+E-1)(C+E+1)}} = \frac{d}{2}\sqrt{(C+E)^2-1}\,\mathrm{mod}\,\frac{\pi}{2}$$



complexification occurs in a standard way with pair eigenvalues rising from the lower continuum. On the fig. there is *c*-dependence of the lowest pair for fixed *d* = 1.5.

Square well - The Dirac case

- One dimensional Dirac Hamiltonian with square-well potential

$$\mathsf{H} = \begin{pmatrix} M + C\chi_{(0,d)}(x) & -\mathrm{i}\partial_x \\ -\mathrm{i}\partial_x & -M + C\chi_{(0,d)}(x) \end{pmatrix}$$

$$\mathrm{Dom}\,\mathsf{H} = \{\psi \in L_2(\mathbb{R}) \oplus L_2(\mathbb{R}) | \psi' \in L_2(\mathbb{R}) \oplus L_2(\mathbb{R})\}$$

- After scaling out the mass M to be equal to 1, the energies are solutions of the equations

$$(1+\xi)\mathrm{e}^{\omega d} = 1 - \xi$$

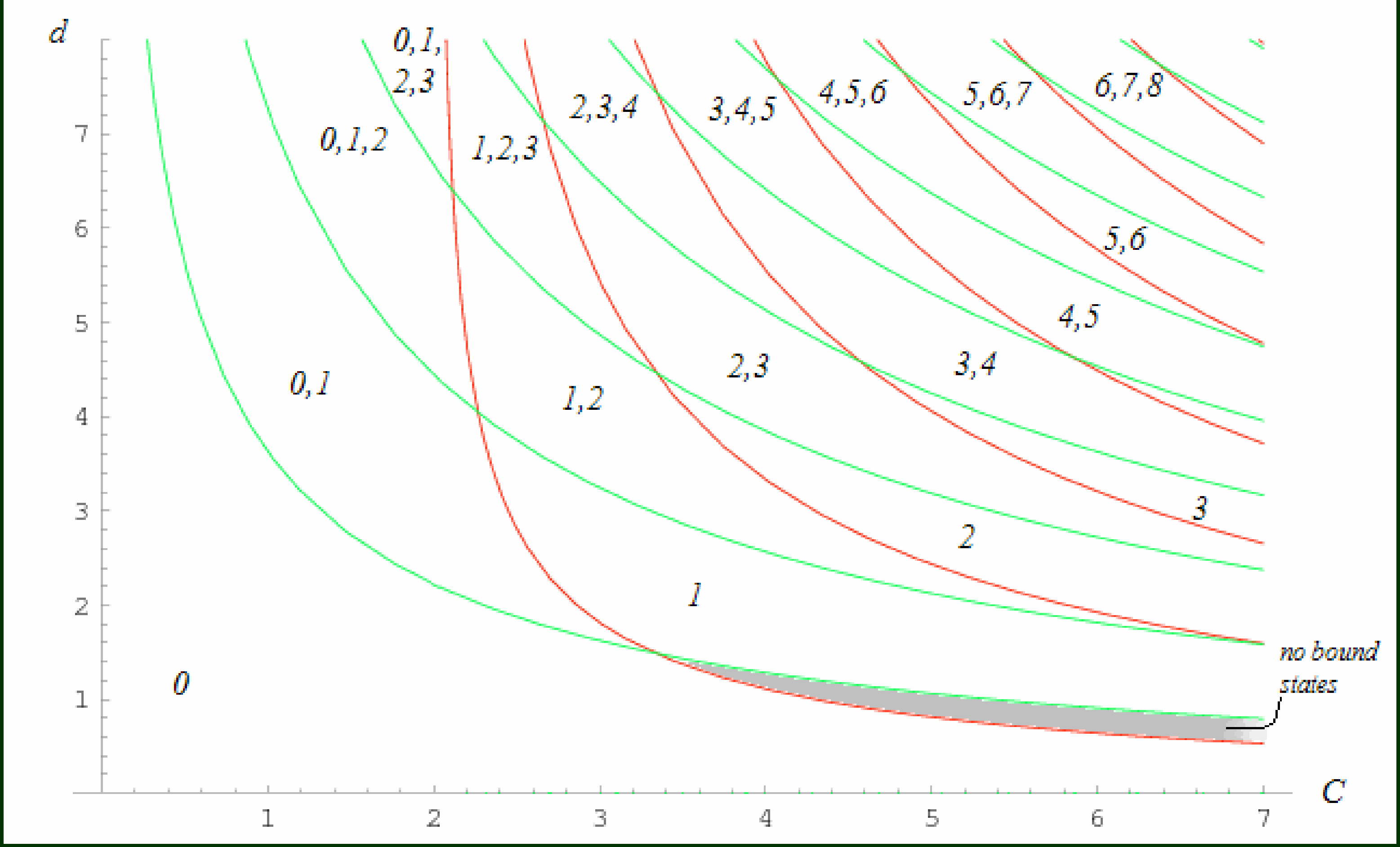
$$\begin{aligned} \rho_\pm &= \sqrt{1 \pm (C+E)} \\ P_\pm &= \sqrt{1 \pm E} \\ \omega &= \rho_+ \rho_- \\ \xi &= \frac{P_- \rho_+}{P_+ \rho_-} \end{aligned}$$

ξ is purely imaginary if and only if ω is, i.e. the energy lies between 1 and 1-C for $C > 0$. The charge-conjugation symmetry for $C \to -C$, $E \to -E$ exists.

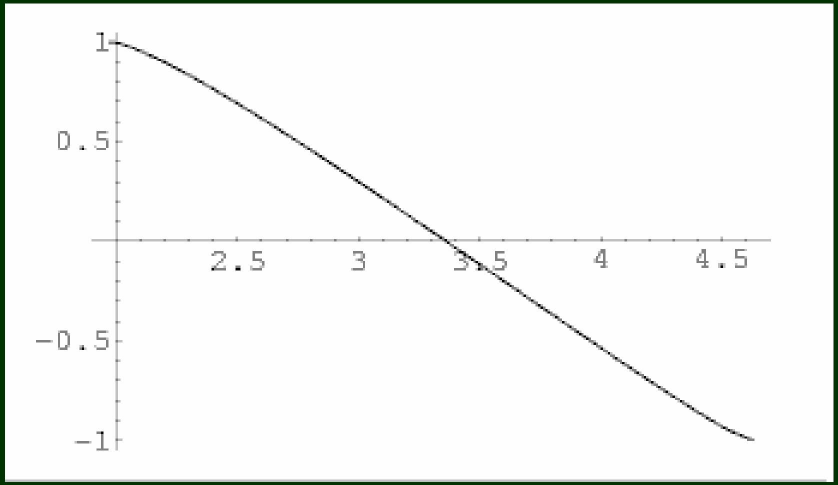
- We get a real equation

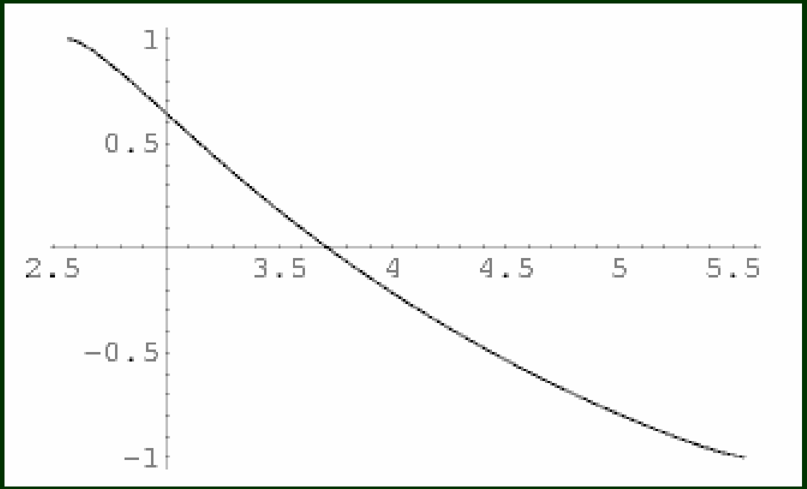
$$\frac{\pi}{2} - \arctan\sqrt{\frac{(1+E)(C+E-1)}{(1-E)(C+E+1)}} = \frac{d}{2}\sqrt{(C+E)^2-1} \bmod \pi$$

- Number of existing eigenvalues for particular C, d can be deduced from above, one gets following picture (eigenstates labelled 0, 1, 2 ...)



- Parametric dependence – the eigenvalues emerge from the upper continuum and sink eventually in the lower as the width and depth rise, but only if the depth $|C| > 2$.
- Typical behaviour of the eigenvalues, *c*- and *d*- dependence:





- No square-root singularity!
- No complex eigenvalues for strong coupling!

- Limit case – delta interaction (Dirac)

$$c \to \infty, \;\; cd \to \text{const.}$$

- only one energy, exactly solvable

$$E = \cos cd$$

- Non-analytic behaviour
- Impossible for KG, analogous limit would result in a trivial system
- For large *c* the KG still maintains square-root EP

$$E = \frac{-c^3 d^2 + 2\sqrt{4 + c^2 d^2 (1 - c^2)}}{4 + c^2 d^2}$$

Summary

- There are different types of complexification.
- Square-root EP is strongly preferred and typical for most Hamiltonians (mechanism similar to the “no crossing theorem”).
- Dirac Hamiltonians can exhibit PT-symmetrically counter-intuitive behaviour.