$\mathcal{PT}$-symmetric non-selfadjoint operators, diagonalizable and non-diagonalizable, with real discrete spectrum

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Recent results with S. Graffi and J. Sjöstrand:

An example of $\mathcal{PT}$-symmetric Schrödinger-type operator with real discrete spectrum which is not diagonalizable is provided. Moreover the class of $\mathcal{PT}$-symmetric Schrödinger operators with real spectrum is enlarged: explicit examples are provided in dimension greater than one, as perturbations of the harmonic oscillators.

Introduction: Two Mathematical Questions in $\mathcal{PT}$-symmetry

A basic fact underlying $\mathcal{PT}$-symmetric quantum mechanics is the existence of non self-adjoint (not even normal) $\mathcal{PT}$-symmetric Schrödinger operators which have fully real spectrum.

In the general case $d \geq 1$ we assign to $\mathcal{P}$ its most general mathematical meaning:

$(\mathcal{P}\psi)(x_1, \dots, x_d) = ((-1)^{j_1}x_1, \dots, (-1)^{j_d}x_d)\,,$

$j_k = 0, 1$

reflexion w.r.t. <u>any</u> subset of the coordinates; $\mathcal{T}$: complex conjugation.

$\mathbf{Q}_1$**)** *<u>Conditions for the reality of the spectrum</u>*

Recall results by:

- Dorey et al. (2002), Shin (2002): ODE;
- C., Graffi, Sjöstrand (2005): <u>Perturbation Th</u>.

Q_2) *Analyze this phenomenon in terms of selfadjoint spectral theory*

If: H is $\mathcal{PT}$-symmetric, $\sigma(H) \subset \mathbf{R}$ and H is diagonalizable

Then: H is conjugate to a selfadjoint operator A: $H = SAS^{-1}$

Question:

H: $\mathcal{PT}$-symmetric Schrödinger-type, $\sigma(H)$ real $\Rightarrow H$ is diagonalizable ?

Answers

(C., Graffi, Sjöstrand, 2007)

R_2) NO:

we provide an explicit simple example of $\mathcal{PT}$-symmetric Schrödinger-type operator H: $\sigma(H)$ real discrete, H is NOT diagonalizable: occurance of **Jordan blocks**

$$H(g) := a_1^* a_1 + a_2^* a_2 + i g a_2^* a_1 + 1, \quad g \in \mathbf{R}, \text{in} L^2(\mathbf{R}^2)$$

where

$$a_i = \frac{1}{\sqrt{2}}\left(x_i + \frac{d}{dx_i}\right), \quad a_i^* = \frac{1}{\sqrt{2}}\left(x_i - \frac{d}{dx_i}\right)$$

standard **destruction** and **creation** operators. Then:

$$H(g) = \frac{1}{2}\left[-\frac{d^2}{dx_1^2} + x_1^2\right] + \frac{1}{2}\left[-\frac{d^2}{dx_2^2} + x_2^2\right]$$

$$+ i g \frac{1}{2}\left(x_2 - \frac{d}{dx_2}\right)\left(x_1 + \frac{d}{dx_1}\right)$$

is invariant under the $\mathcal{PT}$ operation:

$$x_2 \to -x_2\,, \quad ig \to -ig$$

More precisely, $\forall \psi \in L^2(\mathbf{R}^2)$:

$$\mathcal{P}\psi(x_1, x_2) = \psi(x_1, -x_2)\,, \quad \mathcal{T}\psi = \overline{\psi}$$

R$_1$) We identify a new class of non self-adjoint $\mathcal{PT}$-symmetric operators with real spectrum in $L^2(\mathbf{R}^d), d > 1$.

First example in $d > 1$

Example:

Perturbation of harmonic oscillators in $d > 1$

$$H(g) = \frac{1}{2}\sum_{k=1}^{d}\left[-\frac{d^2}{dx_k^2} + \omega_k^2 x_k^2\right] + igW(x_1, \ldots, x_d)\,,$$

$W \in L^\infty(\mathbf{R}^d; \mathbf{R})$, $W(-x_1, \ldots, -x_d) = -W(x_1, \ldots, x_d)$,

$g \in \mathbf{R}$, $|g| < \rho$, $\rho > 0$;

$\omega_k = \dfrac{p_k}{q_k}\omega$: rational multiples of a fixed frequency $\omega > 0$ with:

$p_k, q_k \in \mathbf{N}$ **both odd**, $\forall k = 1, \ldots, d$.

Case $d = 2$

$\sigma(H(g))$ real if and only if p_k, q_k are both odd (also necessary condition!)

$\mathbf{R}_2$) A non diagonalizable $\mathcal{PT}$-symmetric Schrödinger-type operator with real spectrum

$$H(g) = H_0 + igV$$

$$H_0 = \frac{1}{2}\left[-\frac{d^2}{dx_1^2} + x_1^2\right] + \frac{1}{2}\left[-\frac{d^2}{dx_2^2} + x_2^2\right]$$

on $D(H_0) = D(\Delta) \cap D(x_1^2 + x_2^2)$,

$$V = \frac{1}{2}\left(x_2 - \frac{d}{dx_2}\right)\left(x_1 + \frac{d}{dx_1}\right)$$

Theorem 1

For $g \in \mathbf{R}$, $|g| < 2$, $H(g)$ is defined on $D(H_0)$ and:

1) $\sigma(H(g))$ is discrete;

2) the eigenvalues of $H(g)$ are:

$\lambda_n = n + 1, n = 0, 1, 2, \ldots$ *and $\forall n$:*

$m_g(\lambda_n) = 1$ *(geometric multiplicity);*

$m_a(\lambda_n) = n + 1$ *(algebraic multiplicity).*

More precisely:

$$L^2(\mathbf{R}^2) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

$\mathcal{H}_n$ *is invariant under* $H(g)$*;*

$H_n := H(g)|_{\mathcal{H}_n}$, *then*

$$H(g) = \bigoplus_{n=0}^{\infty} H_n$$

and H_n *is represented by the* $(n+1)\times(n+1)$ *matrix:*

$$H_n = (n+1)I_{(n+1)} + igD_n \qquad (1)$$

D_n*: nilpotent of order* $n+1$*:*

$$D_n := \begin{pmatrix} 0 & \sqrt{n} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \sqrt{2(n-1)} & 0 & \cdot & 0 \\ 0 & 0 & 0 & \sqrt{3(n-2)} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{n} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

$\Longrightarrow D_n^{n+1} = 0.$

Remarks

a) $\sigma(H(g)) = \mathbf{N} \setminus \{0\}$ is real and independent of g;

b) (1) is the Jordan canonical form of H_n; algebraic multiplicity $= n + 1$;

$D_n \neq 0 \implies H_n$ is <u>not</u> diagonalizable $\implies$ neither is $H(g)$;

c) $H(g)$ belongs to the class of block-diagonalizable Hamiltonians with finite dimensional diagonal blocks explored by

- A. Mostafazadeh (2002-2004);
- G. Scolarici, L. Solombrino (2003).

Recent investigations on non diagonalizable operators also by A. Andrianov, F. Cannata, A. Sokolov (2007).

Sketch of the proof of the theorem

Step 1

Make use of the Fock-Bargmann representation:

$$U_B : L^2(\mathbf{R}^d) \to \mathcal{F}_d \quad \underline{\text{unitary map}}$$

$\mathcal{F}_d$: space of entire holomorphic functions $f(z) : \mathbf{C}^d \to \mathbf{C}$, $(z = x + iy)$, s.t.

$$\|f(z)\|^2_{\mathcal{F}} := \frac{1}{\pi^d} \int_{\mathbf{R}^{2d}} |f(z)|^2 e^{-|z|^2}\, dx\, dy = \langle f, f\rangle_{\mathcal{F}} < +\infty$$

$\mathcal{F}_d$ is a Hilbert space endowed with the scalar product:

$$\langle f, g\rangle_{\mathcal{F}} = \frac{1}{\pi^d} \int_{\mathbf{R}^{2d}} f(z)\overline{g(z)} e^{-|z|^2}\, dx\, dy$$

Bargmann transform U_B:

$$(U_B\psi)(z) = \frac{1}{\pi^{d/4}} \int_{\mathbf{R}^d} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\langle x, z\rangle + z^2)} \psi(x)\, dx,$$

$\forall \psi \in L^2(\mathbf{R}^d)$, $z \in \mathbf{C}^d$.

Then: $\|U_B\psi\|_{\mathcal{F}} = \|\psi\|_{L^2(\mathbf{R}^d)}$.

Step 2

$$U_B a_i^* U_B^{-1} = z_i, \quad U_B a_i U_B^{-1} = \frac{\partial}{\partial z_i}$$

$$U_B N^{(d)} U_B^{-1} = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i}$$

$N^{(d)} := \sum_{i=1}^{d} N_i$ the total number operator;
$N_i := a_i^* a_i$ i-th number operator.

Then:

$$H_0 = \sum_{i=1}^{2} a_i^* a_i + 1 = N^{(2)} + 1\,;$$

$$H(g) = N^{(2)} + 1 + iga_2^* a_1$$

and

$$Q(g) := U_B(H(g)-1)U_B^{-1} = U_B(N^{(2)}+iga_2^* a_1)U_B^{-1}$$

$$= z_1 \tfrac{\partial}{\partial z_1} + z_2 \tfrac{\partial}{\partial z_2} + igz_2 \tfrac{\partial}{\partial z_1} := Q_0 + igW$$

Remark

$\sigma(Q_0) = \sigma(N^{(2)}) = \{0, 1, \ldots, n, \ldots\}$ and $m_g(n) = m_a(n) = n + 1$.

Step 3

$\psi_k(x)$: k-th normalized eigenvector of the one-dimensional harmonic oscillator in $L^2(\mathbf{R})$. ($\psi_0(x) = \pi^{-1/4} e^{-x^2/2}$)

Then:

$$(U_B \psi_k)(z) := e_k(z) = \frac{z^k}{\sqrt{k!}}, \quad k = 0, 1, \ldots$$

$$f_{n,h}(z_1, z_2) := e_{n-h}(z_2) e_h(z_1), \; h = 0, \ldots, n$$

Set $\quad \mathcal{K}_n := \mathrm{Span}\{f_{n,h} : h = 0, \ldots, n\}$

$$= \mathrm{Span}\{e_{l_1}(z_2) e_{l_2}(z_1) : l_1 + l_2 = n\}$$

Then:

$$\dim \mathcal{K}_n = n + 1; \quad \mathcal{K}_n \perp \mathcal{K}_l, \; n \neq l;$$

$$\bigoplus_{n=0}^{\infty} \mathcal{K}_n = \mathcal{F}_2$$

Lemma2

1) For any $n = 0, 1, \ldots$ and $h = 0, \ldots, n$:

$$Q(g) f_{n,h} = n f_{n,h} + ig\sqrt{h(n-h+1)} f_{n,h-1} \,.$$

2) $\mathcal{K}_n$ reduces $Q(g)$: $Q(g)\mathcal{K}_n \subset \mathcal{K}_n$.

Set $Q(g)_n := Q(g)|_{\mathcal{K}_n}$. Then:

$$Q(g) = \bigoplus_{n=0}^{\infty} Q(g)_n$$

and $Q(g)_n = nI_{(n+1)} + igD_n$ with:

$$D_n := \begin{pmatrix} 0 & \sqrt{n} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \sqrt{2(n-1)} & 0 & \cdot & 0 \\ 0 & 0 & 0 & \sqrt{3(n-2)} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{n} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

Then:

$$L^2(\mathbf{R}^2) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \,, \quad \mathcal{H}_n = U_B^{-1}\mathcal{K}_n$$

$$H(g) = \bigoplus_{n=0}^{\infty} H_n \,, \quad H_n = U_B(Q(g)_n + 1)U_B^{-1}$$

More precisely, to obtain 1):

$$Q(g) f_{n,h} = n f_{n,h} + ig\sqrt{h(n-h+1)} f_{n,h-1}$$

compute the action of

$Q(g) = Q_0 + igW$ on $f_{n,h}$:

$$Q_0 f_{n,h} = (z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}) e_{n-h}(z_2) e_h(z_1)$$

$$= (n-h) e_{n-h}(z_2) e_h(z_1) + h e_{n-h}(z_2) e_h(z_1)$$

$$= n f_{n,h}$$

since

$$z_1 \frac{\partial}{\partial z_1} e_h(z_1) = z_1 \frac{\partial}{\partial z_1} \frac{z_1^h}{\sqrt{h!}} = h \frac{z_1^h}{\sqrt{h!}} = h e_h(z_1)$$

while

$$W f_{n,h} = z_2 \frac{\partial}{\partial z_1} e_{n-h}(z_2) e_h(z_1)$$

$$= z_2 \frac{z_2^{(n-h)}}{\sqrt{(n-h)!}} \frac{\partial}{\partial z_1} \frac{z_1^h}{\sqrt{h!}}$$

$$= h \frac{z_2^{n-(h-1)}}{\sqrt{(n-h)!}} \frac{z_1^{(h-1)}}{\sqrt{h!}} = \sqrt{h(n-h+1)} f_{n,h-1}.$$

R$_1$) A class of non selfadjoint $\mathcal{PT}$-symmetric operators with real spectrum

Assumptions

- H_0: selfadjoint in $L^2(\mathbf{R}^d), d \geq 1$, bounded below (positive), with compact resolvents ($\Rightarrow$ discrete spectrum: $0 \leq \lambda_1 < \lambda_2 < \dots$: increasing sequence of eigenvalues),

H_0 is $\mathcal{P}$-symmetric:

$$\mathcal{P}H_0 = H_0\mathcal{P}$$

-$\mathcal{P}$: standard parity operator defined by:

$$(\mathcal{P}\psi)(x) = \psi(-x)\,, \quad \psi \in L^2(\mathbf{R}^d)\,,\ x \in \mathbf{R}^d$$

and H_0 is also $\mathcal{T}$-symmetric:

$\overline{H_0\psi} = H_0\overline{\psi}$;

-m_r: multiplicity of λ_r;

-$\psi_{r,s}, s = 1, \dots, m_r$: linearly independent eigenfunctins of λ_r;

-$\mathcal{M}_r :=$ Span$\{\psi_{r,s} : s = 1, ..., m_r\}$: eigenspace of λ_r.

Definitions

1) $\mathcal{M}_r$ is even (odd) if the basis vectors $\psi_{r,s}$ are all even (odd):

$\mathcal{P}\psi_{r,s} = \psi_{r,s}, \forall s = 1, \dots, m_r$

($\mathcal{P}\psi_{r,s} = -\psi_{r,s}, \forall s = 1, \dots, m_r$).

2) λ_r is even (odd) if $\mathcal{M}_r$ is even (odd).

Let $W \in L^\infty(\mathbf{R}^d)$ be an odd real function, i.e.

$W(x) = -W(-x)$, $\forall x \in \mathbf{R}^d$.

Then

$$H(g) = H_0 + igW$$

on $D(H_0)$ is $\mathcal{PT}$-symmetric for all $g \in \mathbf{R}$ and has discrete spectrum.

Theorem 3

Under the above assumptions assume further:

(A_1) $\delta := \frac{1}{2}\inf_r(\lambda_{r+1} - \lambda_r) > 0$;

(A_2) $\forall r, \lambda_r$ *is even or odd.*

Then: for $|g| < \frac{\delta}{\|W\|_\infty}$, *the spectrum of* $H(g)$ *is* **purely real**.

In 2005 we proved the result for $d = 1$ and announced the generalization for $d > 1$. Recently (2007) we have proved the result in the general case $d \geq 1$ and we have provided a class of examples for H_0, i.e. conditions for (A_1) and (A_2) to hold.

Theorem4

Let

$$H_0 = \frac{1}{2}\sum_{k=1}^{d}\left[-\frac{d^2}{dx_k^2} + \omega_k^2 x_k^2\right]$$

$$\omega_k = \frac{p_k}{q_k}\omega, \quad k = 1, \ldots, d$$

p_k, q_k: *relatively prime natural numbers.*

Then <u>*(A_1) holds*</u>*. Moreover:*

(i) *If* p_k *and* q_k *are* **both odd**, *for all* k

$\Rightarrow$ <u>*(A_2) holds*</u>*;*

(ii) *If* $d = 2$, *(A_2) holds* **if and only if** p_k *and* q_k *are both odd,* $\forall k$.

Proposition

H_0 *satisfies (A_2)* <u>*if and only if*</u>*:*

$\forall \vec{k} \in \mathbf{Z}^d \setminus \{0\}$ *s. t. the* k_i, $i = 1, \ldots, d$, *have* <u>*no*</u> *common divisors and*

$\langle \vec{\omega}, \vec{k}\rangle := \omega_1 k_1 + \ldots + \omega_d k_d = 0$ (resonant index)

the number $O(\vec{k})$ *of odd* k_i *is* **even**.