PT-symmetric non-selfadjoint operators, diagonalizable and non-diagonalizable, with real discrete spectrum

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Recent results with S. Graffi and J. Sjöstrand:

An example of PT-symmetric Schrödinger-type operator with real discrete spectrum which is not diagonalizable is provided. Moreover the class of PT-symmetric Schrödinger operators with real spectrum is enlarged: explicit examples are provided in dimension greater than one, as perturbations of the harmonic oscillators.
Introduction: Two Mathematical Questions in $\mathcal{PT}$-symmetry

A basic fact underlying $\mathcal{PT}$-symmetric quantum mechanics is the existence of non self-adjoint (not even normal) $\mathcal{PT}$-symmetric Schrödinger operators which have fully real spectrum.

In the general case $d \geq 1$ we assign to $\mathcal{P}$ its most general mathematical meaning:

$$(\mathcal{P}\psi)(x_1, \ldots, x_d) = ((-1)^{j_1}x_1, \ldots, (-1)^{j_d}x_d),$$

$j_k = 0, 1$

reflexion w.r.t. any subset of the coordinates; $\mathcal{T}$: complex conjugation.

$\textbf{Q}_1$) Conditions for the reality of the spectrum

Recall results by:

- Dorey et al. (2002), Shin (2002): ODE;
- C., Graffi, Sjöstrand (2005): Perturbation Th.
\( Q_2 \) Analyze this phenomenon in terms of selfadjoint spectral theory

If: \( H \) is \( \mathcal{PT} \)-symmetric, \( \sigma(H) \subset \mathbb{R} \) and \( H \) is diagonalizable

Then: \( H \) is conjugate to a selfadjoint operator \( A: \quad H = SAS^{-1} \)

**Question:**

\( H \): \( \mathcal{PT} \)-symmetric Schrödinger-type, \( \sigma(H) \) real \( \Rightarrow H \) is diagonalizable?

**Answers**

(C., Graffi, Sjöstrand, 2007)

\( R_2 \) NO:

we provide an explicit simple example of \( \mathcal{PT} \)-symmetric Schrödinger-type operator \( H: \sigma(H) \) real discrete, \( H \) is NOT diagonalizable: occurrence of Jordan blocks
\[ H(g) := a_1^* a_1 + a_2^* a_2 + ig a_2^* a_1 + 1, \quad g \in \mathbb{R}, \text{in } L^2(\mathbb{R}^2) \]

where

\[ a_i = \frac{1}{\sqrt{2}} \left( x_i + \frac{d}{dx_i} \right), \quad a_i^* = \frac{1}{\sqrt{2}} \left( x_i - \frac{d}{dx_i} \right) \]

standard destruction and creation operators. Then:

\[
H(g) = \frac{1}{2} \left[ -\frac{d^2}{dx_1^2} + x_1^2 \right] + \frac{1}{2} \left[ -\frac{d^2}{dx_2^2} + x_2^2 \right] \\
+ ig \frac{1}{2} \left( x_2 - \frac{d}{dx_2} \right) \left( x_1 + \frac{d}{dx_1} \right)
\]

is invariant under the \( \mathcal{PT} \) operation:

\[ x_2 \rightarrow -x_2, \quad ig \rightarrow -ig \]

More precisely, \( \forall \psi \in L^2(\mathbb{R}^2) \):

\[ \mathcal{P} \psi(x_1, x_2) = \psi(x_1, -x_2), \quad T \psi = \overline{\psi} \]
We identify a new class of non self-adjoint $\mathcal{PT}$-symmetric operators with real spectrum in $L^2(\mathbb{R}^d), d > 1$.

First example in $d > 1$

**Example:**

Perturbation of harmonic oscillators in $d > 1$

$$H(g) = \frac{1}{2} \sum_{k=1}^{d} \left[ -\frac{d^2}{dx_k^2} + \omega_k^2 x_k^2 \right] + igW(x_1, \ldots, x_d),$$

$W \in L^\infty(\mathbb{R}^d; \mathbb{R}), W(-x_1, \ldots, -x_d) = -W(x_1, \ldots, x_d),$

$g \in \mathbb{R}, \ |g| < \rho, \ \rho > 0;$

$\omega_k = \frac{p_k}{q_k} \omega$: rational multiples of a fixed frequency $\omega > 0$ with:

$p_k, q_k \in \mathbb{N}$ **both odd**, $\forall k = 1, \ldots, d.$

**Case $d = 2$**

$\sigma(H(g))$ real if and only if $p_k, q_k$ are both odd (also necessary condition!)
A non diagonalizable $\mathcal{PT}$-symmetric Schrödinger-type operator with real spectrum

$$H(g) = H_0 + igV$$

$$H_0 = \frac{1}{2} \left[-\frac{d^2}{dx_1^2} + x_1^2\right] + \frac{1}{2} \left[-\frac{d^2}{dx_2^2} + x_2^2\right]$$
on $D(H_0) = D(\Delta) \cap D(x_1^2 + x_2^2)$,

$$V = \frac{1}{2} \left(x_2 - \frac{d}{dx_2}\right) \left(x_1 + \frac{d}{dx_1}\right)$$

**Theorem 1**

For $g \in \mathbb{R}, |g| < 2$, $H(g)$ is defined on $D(H_0)$ and:

1) $\sigma(H(g))$ is discrete;

2) the eigenvalues of $H(g)$ are:

$$\lambda_n = n + 1, n = 0, 1, 2, \ldots$$

and $\forall n$:

$$m_g(\lambda_n) = 1 \ (\text{geometric multiplicity});$$

$$m_a(\lambda_n) = n + 1 \ (\text{algebraic multiplicity}).$$
More precisely:

\[ L^2(\mathbb{R}^2) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \]

\( \mathcal{H}_n \) is invariant under \( H(g) \);

\( H_n := H(g)\big|_{\mathcal{H}_n} \), then

\[ H(g) = \bigoplus_{n=0}^{\infty} H_n \]

and \( H_n \) is represented by the \((n+1) \times (n+1)\) matrix:

\[ H_n = (n + 1)I_{(n+1)} + igD_n \quad (1) \]

\( D_n: \) nilpotent of order \( n + 1 \):

\[
D_n := \begin{pmatrix}
0 & \sqrt{n} & 0 & \cdots & \cdots & 0 \\
0 & 0 & \sqrt{2(n-1)} & 0 & \cdots & 0 \\
0 & 0 & 0 & \sqrt{3(n-2)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & & \cdots & \sqrt{n} \\
0 & 0 & \cdots & & \cdots & 0
\end{pmatrix}
\]

\[ \Rightarrow D_n^{n+1} = 0. \]
Remarks

a) \( \sigma(H(g)) = \mathbb{N} \setminus \{0\} \) is real and independent of \( g \);

b) (1) is the Jordan canonical form of \( H_n \);
algebraic multiplicity = \( n + 1 \);
\( D_n \neq 0 \implies H_n \text{ is not diagonalizable} \implies \text{neither is } H(g) \);

c) \( H(g) \) belongs to the class of block-diagonalizable Hamiltonians with finite dimensional diagonal blocks explored by
- A. Mostafazadeh (2002-2004);
Recent investigations on non diagonalizable operators also by A. Andrianov, F. Cannata, A. Sokolov (2007).
Sketch of the proof of the theorem

Step 1

Make use of the Fock-Bargmann representation:

\[ U_B : L^2(\mathbb{R}^d) \rightarrow \mathcal{F}_d \quad \text{unitary map} \]

\( \mathcal{F}_d \): space of entire holomorphic functions

\( f(z) : \mathbb{C}^d \rightarrow \mathbb{C}, \ (z = x + iy), \) s.t.

\[ \| f(z) \|_{\mathcal{F}}^2 := \frac{1}{\pi^d} \int_{\mathbb{R}^{2d}} |f(z)|^2 e^{-|z|^2} \, dx \, dy = \langle f, f \rangle_{\mathcal{F}} < +\infty \]

\( \mathcal{F}_d \) is a Hilbert space endowed with the scalar product:

\[ \langle f, g \rangle_{\mathcal{F}} = \frac{1}{\pi^d} \int_{\mathbb{R}^{2d}} f(z) \overline{g(z)} e^{-|z|^2} \, dx \, dy \]

Bargmann transform \( U_B \):

\[ (U_B \psi)(z) = \frac{1}{\pi^{d/4}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \left( x^2 - 2\sqrt{2} \langle x, z \rangle + z^2 \right)} \psi(x) \, dx, \]

\( \forall \psi \in L^2(\mathbb{R}^d), \ z \in \mathbb{C}^d. \)

Then: \( \| U_B \psi \|_{\mathcal{F}} = \| \psi \|_{L^2(\mathbb{R}^d)}. \)
Step 2

\[ U_B a_i^* U_B^{-1} = z_i, \quad U_B a_i U_B^{-1} = \frac{\partial}{\partial z_i} \]

\[ U_B N^{(d)} U_B^{-1} = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i} \]

\[ N^{(d)} := \sum_{i=1}^{d} N_i \] the total number operator;
\[ N_i := a_i^* a_i \] i-th number operator.

Then:

\[ H_0 = \sum_{i=1}^{2} a_i^* a_i + 1 = N^{(2)} + 1 ; \]

\[ H(g) = N^{(2)} + 1 + i g a_2^* a_1 \]

and

\[ Q(g) := U_B (H(g) - 1) U_B^{-1} = U_B (N^{(2)} + i g a_2^* a_1) U_B^{-1} \]

\[ = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + i g z_2 \frac{\partial}{\partial z_1} := Q_0 + i g W \]
Remark

$\sigma(Q_0) = \sigma(N^{(2)}) = \{0, 1, \ldots, n, \ldots\}$ and

$m_g(n) = m_a(n) = n + 1.$

**Step 3**

$\psi_k(x)$: $k$-th normalized eigenvector of the one-dimensional harmonic oscillator in $L^2(\mathbb{R})$. ($\psi_0(x) = \pi^{-1/4} e^{-x^2/2}$)

Then:

$$(U_B\psi_k)(z) := e_k(z) = \frac{z^k}{\sqrt{k!}}, \quad k = 0, 1, \ldots$$

$$f_{n,h}(z_1, z_2) := e_{n-h}(z_2)e_h(z_1), \quad h = 0, \ldots, n$$

Set

$$\mathcal{K}_n := \text{Span}\{f_{n,h} : h = 0, \ldots, n\}$$

$$= \text{Span}\{e_{l_1}(z_2)e_{l_2}(z_1) : l_1 + l_2 = n\}$$

Then:

$$\dim \mathcal{K}_n = n + 1; \quad \mathcal{K}_n \perp \mathcal{K}_l, \quad n \neq l;$$

$$\bigoplus_{n=0}^{\infty} \mathcal{K}_n = \mathcal{F}_2$$
Lemma 2

1) For any \( n = 0, 1, \ldots \) and \( h = 0, \ldots, n \):

\[
Q(g)f_{n,h} = nf_{n,h} + ig\sqrt{h(n-h+1)}f_{n,h-1}.
\]

2) \( \mathcal{K}_n \) reduces \( Q(g) \): \( Q(g)\mathcal{K}_n \subset \mathcal{K}_n \).

Set \( Q(g)_n := Q(g)|_{\mathcal{K}_n} \). Then:

\[
Q(g) = \bigoplus_{n=0}^{\infty} Q(g)_n
\]

and \( Q(g)_n = nI_{(n+1)} + igD_n \) with:

\[
D_n := \begin{pmatrix}
0 & \sqrt{n} & 0 & \cdot & \cdot & 0 \\
0 & 0 & \sqrt{2(n-1)} & 0 & \cdot & 0 \\
0 & 0 & 0 & \sqrt{3(n-2)} & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{n} \\
0 & 0 & \cdot & \cdot & \cdot & 0
\end{pmatrix}
\]

Then:

\[
L^2(\mathbb{R}^2) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_n = U_B^{-1}\mathcal{K}_n
\]

\[
H(g) = \bigoplus_{n=0}^{\infty} H_n, \quad H_n = U_B(Q(g)_n + 1)U_B^{-1}
\]
More precisely, to obtain 1):

\[ Q(g)f_{n,h} = nf_{n,h} + ig\sqrt{h(n-h+1)}f_{n,h-1} \]

compute the action of

\[ Q(g) = Q_0 + igW \] on \( f_{n,h} \):

\[
Q_0f_{n,h} = (z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2})e_{n-h}(z_2)e_h(z_1)
\]

\[ = (n-h)e_{n-h}(z_2)e_h(z_1) + he_{n-h}(z_2)e_h(z_1) = nf_{n,h} \]

since

\[ z_1\frac{\partial}{\partial z_1}e_h(z_1) = z_1\frac{\partial}{\partial z_1}\frac{z_1^h}{\sqrt{h!}} = h\frac{z_1^h}{\sqrt{h!}} = he_h(z_1) \]

while

\[
Wf_{n,h} = z_2\frac{\partial}{\partial z_1}e_{n-h}(z_2)e_h(z_1)
\]

\[ = z_2\frac{z_2^{(n-h)}}{\sqrt{(n-h)!}}\frac{\partial}{\partial z_1}\frac{z_1^h}{\sqrt{h!}} \]

\[ = h\frac{z_2^{n-(h-1)}z_1^{(h-1)}}{\sqrt{(n-h)!}\sqrt{h!}} = \sqrt{h(n-h+1)}f_{n,h-1}. \]
R₁) A class of non selfadjoint $\mathcal{PT}$-symmetric operators with real spectrum

Assumptions
- $H₀$: selfadjoint in $L²(\mathbb{R}^d), d ≥ 1$, bounded below (positive), with compact resolvents ($⇒$ discrete spectrum: $0 ≤ \lambda₁ < \lambda₂ < \ldots$: increasing sequence of eigenvalues),
- $H₀$ is $\mathcal{P}$-symmetric: 
  \[ \mathcal{P}H₀ = H₀\mathcal{P} \]
- $\mathcal{P}$: standard parity operator defined by:
  \[ (\mathcal{P}\psi)(x) = \psi(-x), \quad \psi ∈ L²(\mathbb{R}^d), \quad x ∈ \mathbb{R}^d \]
  and $H₀$ is also $\mathcal{T}$-symmetric:
  \[ \overline{H₀\psi} = H₀\overline{\psi}; \]
- $m_r$: multiplicity of $\lambda_r$;
- $ψ_{r,s}, s = 1, \ldots, m_r$: linearly independent eigenfunctions of $\lambda_r$;
- $M_r := \text{Span}\{ψ_{r,s} : s = 1, \ldots, m_r\}$: eigenspace
Definitions

1) \( M_r \) is even (odd) if the basis vectors \( \psi_{r,s} \) are all even (odd):

\[
P \psi_{r,s} = \psi_{r,s}, \forall s = 1, \ldots, m_r
\]

\[
( P \psi_{r,s} = -\psi_{r,s}, \forall s = 1, \ldots, m_r).
\]

2) \( \lambda_r \) is even (odd) if \( M_r \) is even (odd).

Let \( W \in L^\infty(\mathbb{R}^d) \) be an odd real function, i.e.

\[
W(x) = -W(-x), \forall x \in \mathbb{R}^d.
\]

Then

\[
H(g) = H_0 + igW
\]
on \( D(H_0) \) is \( \mathcal{PT} \)-symmetric for all \( g \in \mathbb{R} \) and has discrete spectrum.
Theorem 3

Under the above assumptions assume further:

(A₁) $\delta := \frac{1}{2}\inf_r(\lambda_{r+1} - \lambda_r) > 0$;

(A₂) $\forall r, \lambda_r$ is even or odd.

Then: for $|g| < \frac{\delta}{\|W\|_\infty}$, the spectrum of $H(g)$ is purely real.

In 2005 we proved the result for $d = 1$ and announced the generalization for $d > 1$.

Recently (2007) we have proved the result in the general case $d \geq 1$ and we have provided a class of examples for $H_0$, i.e. conditions for (A₁) and (A₂) to hold.
Theorem 4

Let

\[ H_0 = \frac{1}{2} \sum_{k=1}^{d} \left[ -\frac{d^2}{dx_k^2} + \omega_k^2 x_k^2 \right] \]

\[ \omega_k = \frac{p_k}{q_k} \omega, \quad k = 1, \ldots, d \]

\( p_k, q_k \): relatively prime natural numbers.

Then (A₁) holds. Moreover:

(i) If \( p_k \) and \( q_k \) are both odd, for all \( k \)

\( \Rightarrow \) (A₂) holds;

(ii) If \( d = 2 \), (A₂) holds if and only if \( p_k \) and \( q_k \) are both odd, \( \forall k \).

Proposition

\( H_0 \) satisfies (A₂) if and only if:

\( \forall \vec{k} \in \mathbb{Z}^d \setminus \{0\} \) s. t. the \( k_i, i = 1, \ldots, d \), have no common divisors and

\[ \langle \vec{\omega}, \vec{k} \rangle := \omega_1 k_1 + \ldots + \omega_d k_d = 0 \quad \text{(resonant index)} \]

the number \( O(\vec{k}) \) of odd \( k_i \) is even.