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Indefinite metric Quantization

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Indefinite Metric Quantization

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- Properties of the indefinite metric Fock quantization are precisely studied.
- The presence of negative norm state or negative frequency solution is indispensable for a fully covariant quantization of the minimally coupled scalar field in de sitter space.
- λφ4 theory in Minkowski space is considered
- Turning negative norms to account allows ultraviolet divergence in vacuum energy to be eliminated.

Negative probabilities and the notion of an indefinite metric have repeatedly been considered in physics so far, while no explicit contradictions could be pointed out, it has often been suggested that, introducing an indefinite metric would make quantum theory inconsistent .

1- E.C.G.Sudashan, phys. Rev. 123, No6,(1961)2183

2-Mostafazadeh A., quant-ph/0308028-J. Math.Phys. 43,205 (2002) -J.Math.Phys. 43,2814(2002)

3-P.A.M.Dirac, proc. Roy. Soc A 180 1 (1942)

Indefinite Hilbert space was proposed in 1942 by Dirac. In 1950, Gupta applied it in QED to avoid the negative energy photons

1-Wightman, A.S. Nuovo Cimento Suool. XIV, 81 (1959)

2- Bracci L. , Morchio G. and Strocchi F. , Comm. Math. Phys. 42 , 285 (1975)

3- S. N. Gupta, Proc. Phys. Soc. Sect. A. 63,681 (1950)

The mathematical studies can be found in

J.P.Gazeau,J.Renaud, M.V. Takook, Class. Quantum Grav., 17(2000)1415, gr-qc/9904023

Here we will review some properties of negative norm state appearance in Minkowski and de Sitter space-time. Noting that using **indefinite metric** and **negative norms** which is a different approach in quantum field theory to eliminate the divergence appearance in field theory is consequential;

it would be a **clue** to quantization of gravity.

WHY DE SITTER SPACE?

- **Experimental data has suggested that our universe is a de Sitter universe**
- **The maximal symmetry of de Sitter space**
- **Field equations have the simplest form here**
- **The vacuum state can be investigated and one-particle state can be produced**
- **On the top, this space suggests the inflationary model for our universe**

- De Sitter space
- In mathematics and physics, *n*-dimensional de Sitter space, denoted *dSn*, is the Lorentzian analog of an *n*-sphere. It is a maximally symmetric, Lorentzian manifold with constant positive curvature, and is simply-connected for *n* at least 3.
- In the language of general relativity, de Sitter space is the maximally symmetric, vacuum solution of Einstein's field equation with a positive cosmological constant Λ.

- When $n = 4$, it is also a cosmological model for the physical universe; see de Sitter universe.
- De Sitter space was discovered by Willem de Sitter, and independently by Tullio Levi-Civita (1917).

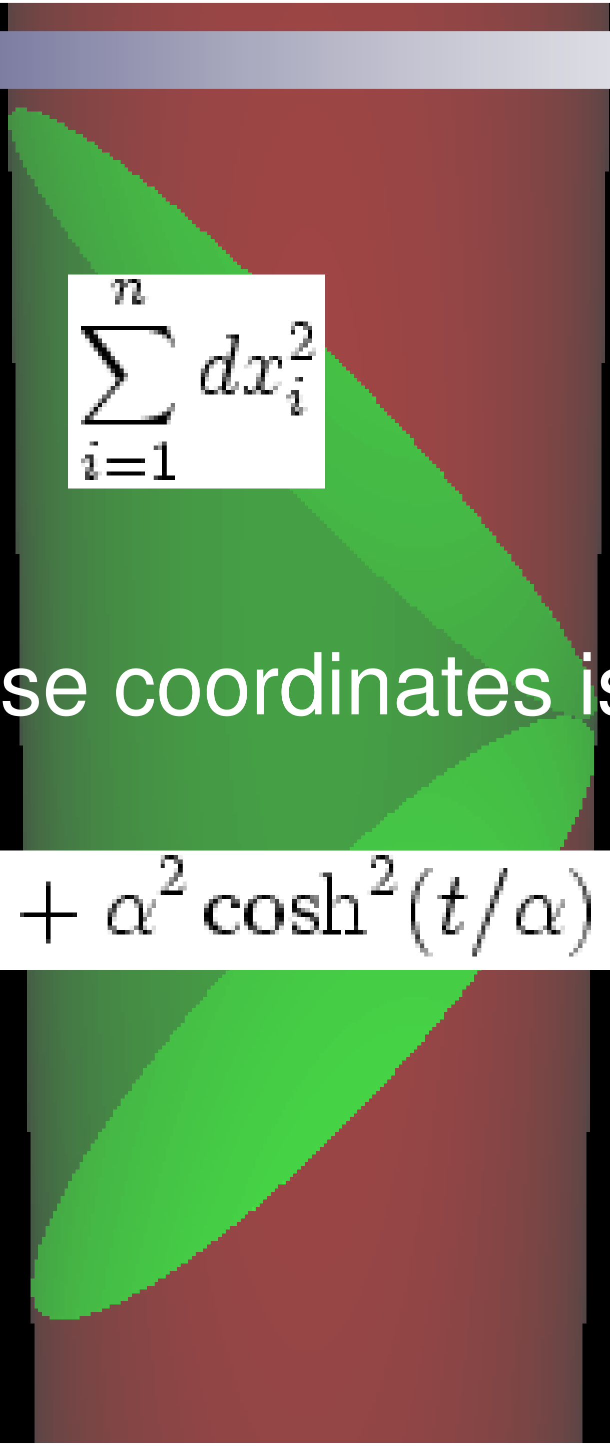
- Definition
- De Sitter space can be defined as a submanifold of Minkowski space in one higher dimension. Take Minkowski space R1,*n* with the standard metric:

$$ds^2 = -dx_0^2 + \sum_{i=1}^{n} dx_i^2.$$

- De Sitter space is the submanifold described by the hyperboloid

$$-x_0^2 + \sum_{i=1}^{n} x_i^2 = \alpha^2$$

α is some positive constant with dimensions of length. The metric on de Sitter space is the metric induced from the ambient Minkowski metric. The induced metric is nondegenerate and has Lorentzian signature. De Sitter space can also be shown that is a non-Riemannian symmetric space.



$$\sum_{i=1}^{n} dx_i^2$$

- metric in these coordinates is given by

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha)\, d\Omega_{n-1}^2$$

- Properties
- The Isometry group of de Sitter space is the Lorentz group O(1,*n*). The metric therefore has *n*(*n*+1)/2 independent Killing vector and is maximally symmetric . Every maximally symmetric space has constant curvature. The Riemann curvature tensor of de Sitter is given by

$$R_{\rho\sigma\mu\nu} = \frac{1}{\alpha^2}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$$

- De Sitter space is an Einstein manifold since the Ricci tensor is proportional to the metric:

$$R_{\mu\nu} = \frac{n-1}{\alpha^2} g_{\mu\nu}$$

- This means de Sitter space is a vacuum solution of Einstein's equation with cosmological constant given by

$$\Lambda = \frac{(n-1)(n-2)}{2\alpha^2}.$$

- The scalar curvature of de Sitter space is given by

$$R = \frac{n(n-1)}{\alpha^2} = \frac{2n}{n-2}\Lambda.$$

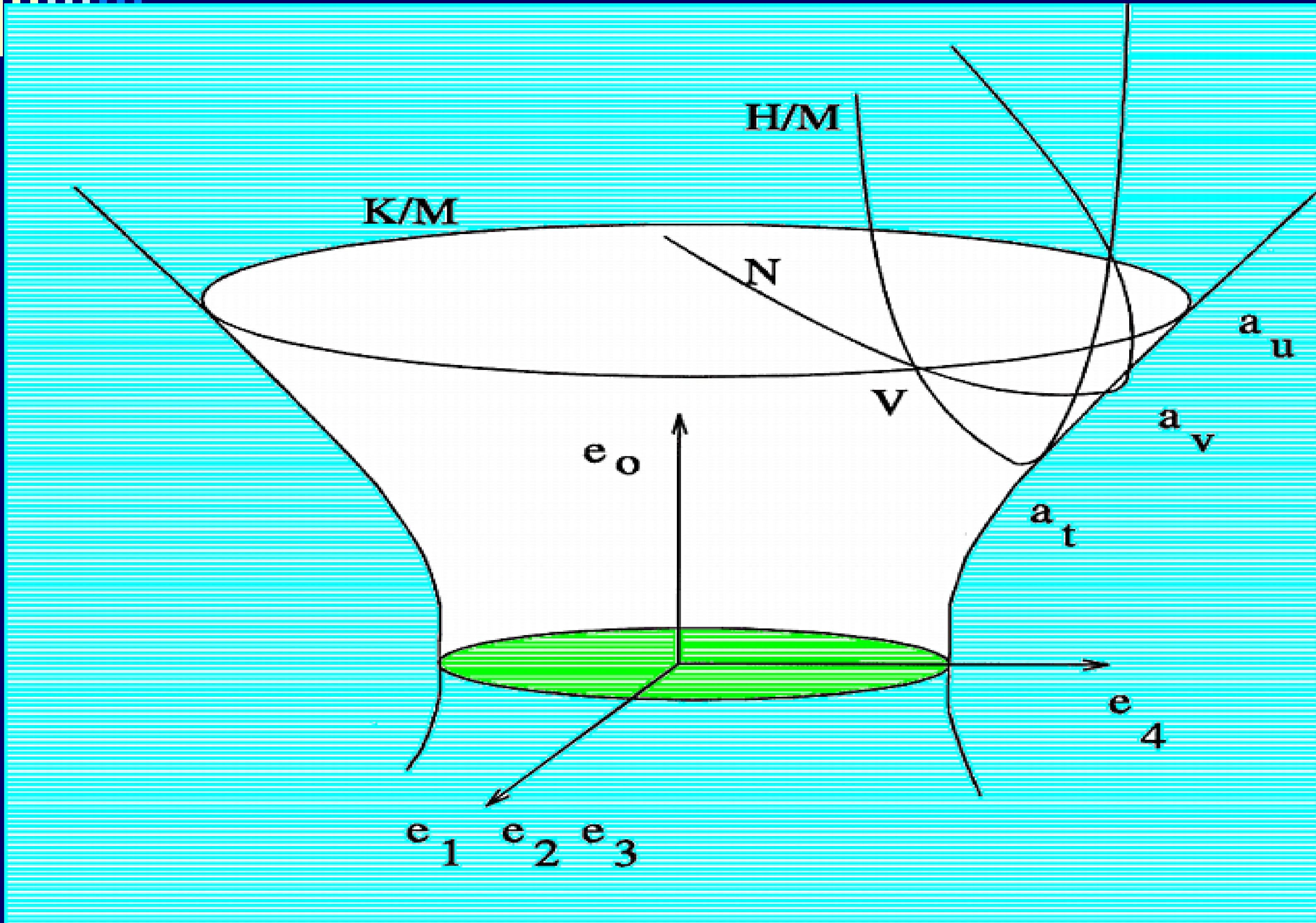
For the case *n* = 4, we have Λ = 3/α2 and *R* = 4Λ = 12/α2.

- A four-dimension de Sitter space can be conveniently defined as a hyperboloid embedding in a five-dimensional Minkowski space

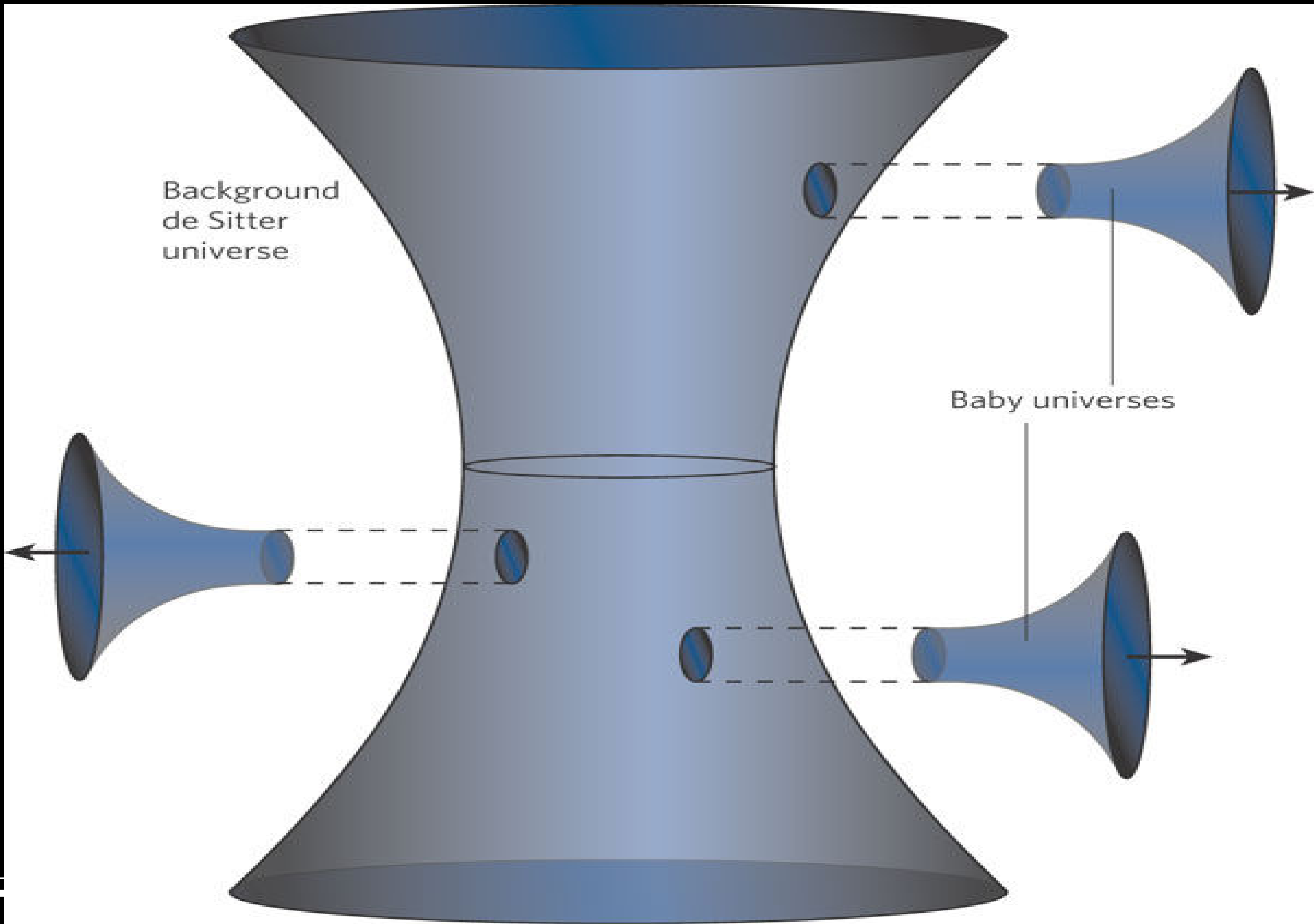
$$g_{\mu\upsilon} x^{\mu} x^{\upsilon} = -H^{-2}$$

$g_{\mu\upsilon} = diag(-1,1,1,1,1)$ and $\mu, \upsilon = 0,1,2,3,4$

H Is Hubble constant



H/M
K/M
N
V
a_u
a_v
a_t
e_o
e_4
e_1 e_2 e_3



Background
de Sitter
universe
Baby universes

- Minimally coupled **massless** scalar field in the de Sitter space described by:

$$\Box_H \Phi(x) = 0$$

$\Box_H$ is the **Laplas-Beltrami** operator in de sitter space

1-Birrel N.D.,Davies P.C.W.,Cambridge University Press, "Quantum Field in Curred space"(1982)

- **Allen proved the covariant canonical quantization procedure with positive modes and the zero modes are not de Sitter invariant, or are not closed under the action of the de Sitter group.**
- **Gazeau et al. proved that in order to obtain a fully covariant quantization of minimally coupled scalar field in de Sitter space, the negative norm states are necessary.**

1-B. Allen, Phys. Rev. D, 32(1985)3136

2-J.P.Gazeau,J.Renaud, M.V. Takook, Class. Quantum Grav.,17(2000)1415, gr-qc/9904023

Hilbert space

$$H = \{ \sum_{k \geq 0} \alpha_k \phi_k , \sum_{k \geq 0} |\alpha_k|^2 \langle \infty \}$$

$$K = \{ (L, l, m) \in N \times N \times Z; 0 \leq l \leq L, -l \leq m \leq l \}$$

In order to obtain a fully covariant quantum field, the conjugate modes are added.

So we have orthogonal sum of a positive and negative inner product spaces.

- negative values of inner product are precisely produced by the conjugate modes

$$\left\langle \phi_k^*, \phi_k^* \right\rangle = -1, k \geq 0$$

the space of solution contains the unphysical states with negative norm.

Indefinite inner product

$$(\phi_1, \phi_2) \quad \rangle 0$$

$$= 0$$

$$\langle 0$$

last one used to be ignored in field theory by physicisist

Why negative values

of

Inner product

used to be ignored in physics for ages?

It had happened without any exact reason

Gupta-Bluler formalism

In electrodynamics the Gupta-Bluler triplet $V_g \subset V \subset V'$ is defined as follows:

the space V_g is the space of longitudinal photon states or "gauge states", the space V is the space of positive frequency solutions of the field equation verifying the Lorentz condition, and V' is the space of all positive frequency solutions of the field equation, containing non-physical states.

The Klein-Gordon inner product defines an indefinite inner product on V' which is Poincare invariant. All three spaces carry representation of the Poincare group but V_g and V are not invariantly complemented.

1-N. Gisin, J. Phys. A 14,2259 (1981)

2-W. Daniel, Helv. Phys. Acta55, 330 (1982)

- The quotient space V/V_g of states up to a gauge transformation is the space of physical one-photon states. The quantized field acts on the Fock space built on V' which is not a Hilbert space, but is instead an indefinite inner-product space.

- The space of solutions contain the unphysical states with negative norms.

$$\phi(x) = \frac{1}{\sqrt{2}}[\phi_p(x) + \phi_n(x)]$$

$$\phi_p(x) = \sum_{k \geq 0}(a_k \phi_k(x) + H.C.)$$

$$\phi_n(x) = \sum_{k \geq 0}(b_k \phi^*(x) + H.C.)$$

H.C. stands for Harmonic Conjugate

$[b_k, b_{k'}^\dagger] = -\delta_{kk'}$

- positive mode $\phi_p(x)$ is scalar field
- It is easy to check

$$a_k |0\rangle = 0 \qquad [a_k, a_{k'}^\dagger] = \delta_{kk'}$$

$$b_k |0\rangle = 0 \qquad [b_k, b_{k'}^\dagger] = -\delta_{kk'}$$

- **It has been shown that by introducing the negative norm states in theory, the “normal ordering” procedure for eliminating the ultra violet divergence in vacuum energy, which appears in usual quantum field theory, is not needed.**

- J.P.Gazeau,J.Renaud, M.V. Takook, Class. Quantum Grav.,
- 17(2000)1415, gr-qc/9904023
- M. V. Takook, gr-qc/0006019

- **Electrodynamics interprets a^{α} and $a^{\dagger\alpha}$ as photon annihilation and creation operators with commutation relation**

$$[a^{\alpha}(\vec{k}), a^{\dagger\beta}(\vec{k}')] = \eta^{\alpha\beta}\delta(\vec{k} - \vec{k}')$$

- a single photon state with polarization *α* and momentum *k* is constructed by applying $a^{\dagger\alpha}(k)$ operator, on vacuum state $|0\rangle$. With ordinary metric like we have

$$\eta^{\alpha\beta} = diag(-1,1,1,1) \qquad \alpha, \beta = 0,1,2,3$$

we have

$$\langle 0 | a^{\alpha}(\vec{k}) a^{\dagger\beta}(\vec{k}^{'}) | 0 \rangle = \eta^{\alpha\beta} \delta(\vec{k} - \vec{k}^{'})$$

- If $\alpha = \beta = 0$ then it would be a

Negative definite quantity

For a deeper discussion see:

1-F. Mandle and G. Shaw ,Quantum Field Theory, wiley,

2-C. Itzykson, J-B. Zuber, McGraw-Hill, Inc. (1988) Quantum Field Theory

- scalar field in Minkowski space satisfies

$$(\Box + \mathrm{m}^2)\phi_{(x)} = 0$$

Klein-Gordon product

$$(\phi_1, \phi_2) = -i\int_{t=cons\tan t} \phi_1(x)\overleftrightarrow{\partial}_t \phi_2^{\,*}(t) d^3x$$

$$\mathrm{u_p}(k, x) = Ce^{-ik.x}$$

$$\mathrm{u_n}(k, x) = Ce^{ik.x}$$

$$C = \frac{1}{\sqrt{(2\pi)^3 2\omega}} \qquad \omega_{(\vec{k})} = k^0 = (k.k + m^2)^{\frac{1}{2}} \geq 0$$

- $u(k, x)$ modes are orthogonal and normalized with respect to defined inner product

$$(\mathrm{u}_{\mathrm{p}}(k, x), \mathrm{u}_{\mathrm{p}}(k', x)) = \delta^3(\vec{k} - \vec{k}')$$

$$(\mathrm{u}_{\mathrm{n}}(k, x), \mathrm{u}_{\mathrm{n}}(k', x)) = -\delta^3(\vec{k} - \vec{k}')$$

$$(\mathrm{u}_{\mathrm{p}}(k, x), \mathrm{u}_{\mathrm{n}}(k', x)) = 0$$

- general classical field solution

$$\phi(x) = \int d^3\vec{k}[a(\vec{k})u_p(k,x) + b(\vec{k})u_n(k,x)]$$

$a(\vec{k})$ and $b(\vec{k})$ are two independent coefficients

- The quantization in Krein space which has been done by Takook allows the quantum field has been defined as

$$\phi_{p}(x) = \frac{1}{\sqrt{2}}[\phi_p(x) + \phi_n(x)]$$

$$\phi_p(x) = \int d^3\vec{k}[a(\vec{k})u_p(k,x) + a^\dagger(\vec{k}')u_p{}^*(k,x)]$$

$$\phi_n(x) = \int d^3\vec{k}[b(\vec{k})u_n(k,x) + b^\dagger(\vec{k}')u_n{}^*(k,x)]$$

$a(\vec{k})$ and $b(\vec{k})$ are two independent operators

Creation and annihilation operators

$$[a(\vec{k}),a(\vec{k}')]=0$$

$$[a^{\dagger}(\vec{k}),a^{\dagger}(\vec{k}')]=0$$

$$[a(\vec{k}),a^{\dagger}(\vec{k}')]=\delta(\vec{k}-\vec{k}')$$

$$[b(\vec{k}),b(\vec{k}')]=0$$

$$[b^{\dagger}(\vec{k}),b^{\dagger}(\vec{k}')]=0$$

$$[b(\vec{k}),b^{\dagger}(\vec{k}')]=-\delta(\vec{k}-\vec{k}')$$

$$[a\ \ (\vec{k}),b(\vec{k}')]=0$$

$$[a^{\dagger}(\vec{k}),b^{\dagger}(\vec{k}')]=0$$

$$[a\ \ (\vec{k}),b^{\dagger}(\vec{k}')]=0$$

$$[a^{\dagger}(\vec{k}),b\ \ (\vec{k}')]=0$$

- vacuum state $|0\rangle$

$$a^\dagger(\vec{k})|0\rangle = |1_{\vec{k}}\rangle \qquad a(\vec{k})|0\rangle = 0 \qquad \forall\vec{k}$$

$$b^\dagger(\vec{k})|0\rangle = |\bar{1}_{\vec{k}}\rangle \qquad b(\vec{k})|0\rangle = 0 \qquad \forall\vec{k}$$

$$b(\vec{k})|1_{\vec{k}}\rangle = 0 \qquad a(\vec{k})|1_{\vec{k}}\rangle = 0 \qquad \forall\vec{k}$$

$|1_{\vec{k}}\rangle$ is called a-one **particle state**

$|\bar{1}_{\vec{k}}\rangle$ is called a-one "**unparticle state**"

$$\langle 0 | 0 \rangle = 1$$

$$\langle 1_{\vec{k}'} | 1_{\vec{k}} \rangle = \delta(\vec{k} - \vec{k}\,')$$

$$\langle \bar{1}_{\vec{k}'} | \bar{1}_{\vec{k}} \rangle = -\delta(\vec{k} - \vec{k}\,')$$

$$H = \int d^3\vec{k} k^0 [a^\dagger(\vec{k}) a(\vec{k}) + b^\dagger(\vec{k}) b(\vec{k}) + a^\dagger(\vec{k}) b^\dagger(\vec{k}) + a(\vec{k}) b(\vec{k})]$$

$$\langle N_{\vec{k}'} | H | N_{\vec{k}'} \rangle = \int \langle N_{\vec{k}'} | a^\dagger(\vec{k}) a(\vec{k}) | N_{\vec{k}'} \rangle k^0 d^3\vec{k} \geq 0 \qquad |0_{\vec{k}'}\rangle \equiv |0\rangle$$

Conclusion

Here, negative norm states were used to obtain vacuum energy in Minkowski space-time which is free of any divergence so we did not need "normal ordering" procedure; furthermore, utilizing this method we reviewed massless minimally coupled scalar field in de Sitter universe. It seems that negative norm states and indefinite metric are totally part of the structure of the renormalized field and there is no need to remove them.

During the recent years, so many efforts have been done, in vain, in order to quantize the gravity, and also many different theories have proceeded this issue nevertheless all were unavailing.

At the mercy of this approach may accomplishing the quantization of gravity be easier, because it was seen that the elimination of infinity which appears in vacuum energy takes place.

Shall we expect

that

this procedure

enables us to

have a new approach to

quantum field theory

for

eliminating the other infinities?

We would like to thank Dr. Takook for his very useful discussions

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