$\mathcal{PT}$ symmetry in multi-dimensional solvable quantum potentials

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What is new in higher-dimensional solvable potentials?

Separation of the radial and angular variables

How to chose the angle-dependent potentials in $d = 2$ and 3 dimensions?

Summary and outlook

Motto: *THOUGH THIS BE MADNESS,*
YET THERE IS METHOD IN'T

W. Shakespeare, *Hamlet*

What is new in higher-dimensional solvable potentials?

	Hermitian		$\mathcal{PT}$-symmetric	
	$d=1$	$d>1$	$d=1$	$d>1$
Spectrum	real	real	real+complex	?
Degeneracy	no	yes	(no)	?
Algebra	yes	yes	yes	?
$\mathcal{PT}$-breaking	—	—	yes	?
Pseudo-norm	—	—	indefinite	?
Quasi-parity	—	—	yes	?

Solvable potentials in $d>1$

Separation of the variables

– Cartesian coordinates

– **Polar coordinates** $\Longrightarrow$ non-central potentials

– Parabolic coordinates

– ...

What is seen from the characteristic features in higher dimensions?

$\mathcal{PT}$-symmetric Hamiltonians in various dimensions

$$H = T + V(\mathbf{r})$$

T is **always** $\mathcal{PT}$-symmetric:

$$T = \frac{\mathbf{p}^2}{2m} = -\frac{\hbar^2}{2m}\Delta \qquad \text{(take ħ = 2m = 1 from now on)}$$

What about $V(\mathbf{r})$? Use polar coordinates $\mathcal{PT}$: $\mathbf{r} \Longrightarrow \ -\mathbf{r}$

$d = 1$ dimension:

$$V(x) = \mathcal{PT}V(x)(\mathcal{PT})^{-1} = V^*(-x)$$

$d = 2$ dimensions:

$$V(\rho, \varphi) = \mathcal{PT}V(\rho, \varphi)(\mathcal{PT})^{-1} = V^*(\rho, \varphi + \pi)$$

$d = 3$ dimensions:

$$V(r, \theta, \varphi) = \mathcal{PT}V(r, \theta, \varphi)(\mathcal{PT})^{-1} = V^*(r, \pi - \theta, \varphi + \pi)$$

Central potentials $V(\mathbf{r}) = V(|\mathbf{r}|)$ are uninteresting: $V(r) = V^*(r)$

But **special** non-central potentials can be interesting:

The typical solutions of centrally symmetric problems reflect $\mathcal{PT}$ symmetry:

$$\mathcal{PT}\exp(\mathrm{i}p\varphi) = (-1)^p \exp(-\mathrm{i}p\varphi)$$

$$\mathcal{PT}Y_{lm}(\theta,\varphi) = (-1)^{l+m} Y_{l-m}(\theta,\varphi)$$

$\mathcal{PT}$ symmetry seems to allow *special* non-central potentials. . .

. . . just like in one dimension

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}-V(\rho,\varphi)\psi+E\psi=0\ .$$

Search for separable variables:

$$\psi(\rho,\varphi)=\rho^{-1/2}\phi(\rho)\tau(\varphi)$$

$$\phi''\tau+\frac{1}{\rho^2}\phi\tau''-\left(V(\rho,\varphi)-\frac{1}{4\rho^2}-E\right)\phi\tau=0\ ,$$

Assume that

$$\boxed{\tau''=(K(\varphi)-k)\tau}$$

Then a radial equation is obtained

$$-\phi''+\left[V_0(\rho)+\left(k-\frac{1}{4}\right)\frac{1}{\rho^2}\right]\phi-E\phi=0\ .$$

where

$$V(\rho,\varphi)=V_0(\rho)+\frac{1}{\rho^2}K(\varphi)$$

$V(\rho,\varphi) = V_0(\rho) + \frac{1}{\rho^2} K(\varphi)$ is $\mathcal{PT}$-symmetric **if**

- $V_0(\rho)$ is **real**
- $K(\varphi)$ is **$\mathcal{PT}$-symmetric:** $K^*(\varphi+\pi) = K(\varphi)$
- k is real/complex $\Longrightarrow$ E is real/complex

spontaneous breakdown of $\mathcal{PT}$ symmetry is possible

How to chose $K(\varphi)$ and k?

$\mathcal{PT}$-symmetric $d = 1$ potentials can be used, **but**

- $K(\varphi)$ must be 2π-periodic
- Its solutions **need not** vanish at the boundaries ($\varphi = 0,\ 2\pi$)
- The $\mathcal{PT}$ condition is now $K^*(\varphi+\pi) = K(\varphi)$, **not** $K^*(-\varphi) = K(\varphi)$
- The energy k of the angular problem appears in the radial problem like an angular momentum

Consider some examples

A simple example with $k = 0$

This is solvable only for the ground state, but illustrates the most important features

$\tau(\varphi) = c_\varphi \exp(\mathrm{i}\sin(p\varphi))$

For $k = 0$ this solves

$$V(\rho,\varphi) = V_0(\rho) - \frac{p^2}{\rho^2}\left[\frac{1}{2}\cos(2p\varphi) + \frac{1}{2} + \mathrm{i}\sin(p\varphi)\right]$$

which is $\mathcal{PT}$-symmetric, **if** p is **odd**

Note the different periodicity of the real and imaginary potential components

The $\mathcal{PT}$-normalization of $\tau(\varphi)$ is

$$c_\varphi = [2\pi J_0(2)]^{-1/2} \qquad \text{(Bessel function)} \qquad \textit{irrespective of } p$$

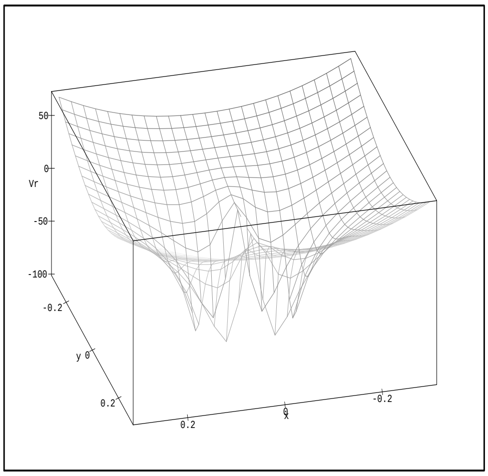
$\mathcal{PT}$ transformation property:

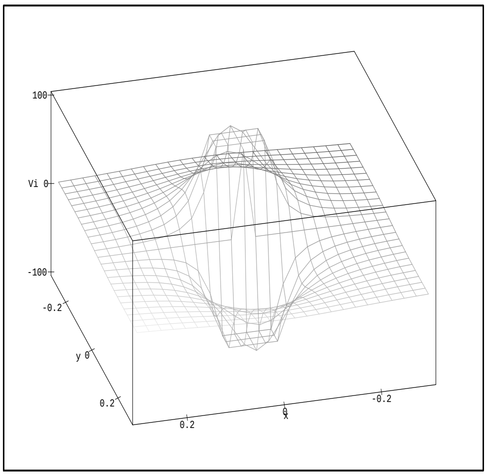
$$\mathcal{PT}\ \tau(\varphi) = \tau(\varphi)$$

Combine this with $V_0(\rho)$ being the H.O., Coulomb, square well, QES, . . . potentials

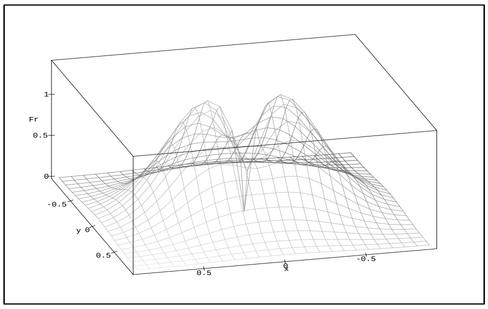
An example: radial harmonic oscillator

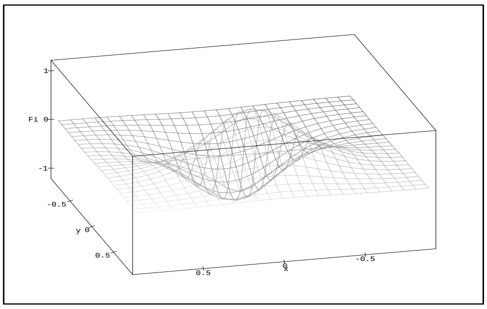
$V_0(\rho) = \omega^2 \rho^2$ with $\omega = 20$

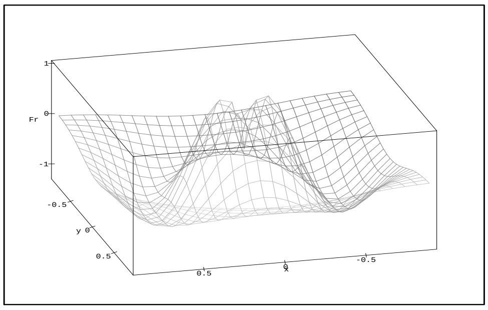


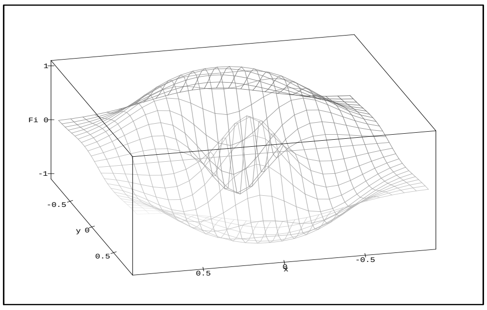


The wavefunctions with $n = 0$ and $n = 1$ belong to $E_0 = \omega = 20$, and $E_1 = 3\omega = 60$









Further choices of fully solvable $K(\varphi)$ potentials

Shape-invariant potentials defined on a finite domain

$\mathcal{PT}$-symmetric Rosen–Morse I and Scarf I (trigonometric) potentials

Consider the Scarf I potential in some detail

G. Lévai, J. Phys. A **39** *(2006) 10161*

$$K(\varphi) = V(\varphi) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)\frac{1}{\cos^2(\varphi)} - \frac{\alpha^2-\beta^2}{2}\frac{\sin(\varphi)}{\cos^2(\varphi)}$$

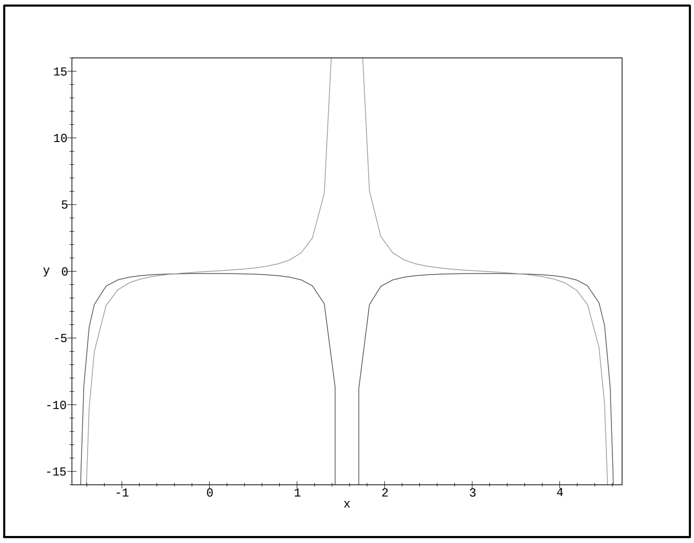
$$\varphi \in (-\pi/2, \pi/2)$$

$$k = E_n = \left(n + \frac{\alpha+\beta+1}{2}\right)^2$$

$$\tau_n(\varphi) = c_n(1-\sin\varphi)^{\frac{\alpha}{2}+\frac{1}{4}}(1+\sin\varphi)^{\frac{\beta}{2}+\frac{1}{4}}P_n^{(\alpha,\beta)}(\sin\varphi)$$

$\alpha = \beta^* = 0.5 + \mathrm{i}\ 0.4$:

weakly singular attractive real potential component; two regular solutions



Note again the different periodicity of the real and imaginary potential components

Main features in $d = 1$:

- $\mathcal{PT}$-symmetric if $\beta^* = \pm\alpha$ $\implies$ real/complex E_n
- Singular at $\varphi = \pm\pi/2$:

$$V(\varphi \to -\frac{\pi}{2}) \sim (\beta^2 - \frac{1}{4})(\pi/2 + \varphi)^{-2} \qquad V(\varphi \to \frac{\pi}{2}) \sim (\alpha^2 - \frac{1}{4})(\pi/2 - \varphi)^{-2}$$

- Two solutions behaving at the boundaries as

$$\lim_{\varphi \to -\pi/2} \psi^{(\pm)}(\varphi) \sim (\pi/2 + \varphi)^{\pm\beta + \frac{1}{2}} \qquad \lim_{\varphi \to \pi/2} \psi^{(\pm)}(\varphi) \sim (\pi/2 - \varphi)^{\pm\alpha + \frac{1}{2}} ,$$

- "quasi"-quasi-parity $q = \pm 1$: $V(\varphi)$ is insensitive to α, $\beta \Longrightarrow -\alpha$, $-\beta$
- **BUT** only one of the solutions is allowed for $|\alpha_R|$, $|\beta_R| > \frac{1}{2}$
- and also for $|\alpha_R|$, $|\beta_R| \leq \frac{1}{2}$ as $\psi^{(+)}$ and $\psi^{(-)}$ are **not** $\mathcal{PT}$-orthogonal
- The sign of the pseudo-norm oscillates as $(-1)^n$
- The spontaneous breakdown of $\mathcal{PT}$ symmetry cannot be defined

Novelties in the $d > 1$ case:

- Extended domain: $\varphi \in [-\pi/2, 3\pi/2]$
- Solutions with finite value at the boundaries are allowed

$\beta^* = \alpha$:   real energies

$$V(\rho,\varphi) = V_0(\rho) + \frac{1}{\rho^2}\left[\left(\alpha_R^2 - \alpha_I^2 - \frac{1}{4}\right)\frac{1}{\cos^2(\varphi)} - 2\mathrm{i}\alpha_R\alpha_I\frac{\sin(\varphi)}{\cos^2(\varphi)}\right]$$

Radial equation:

$$-\phi'' + \left[V_0(\rho) + \frac{(k+\alpha_R+1/2)^2 - 1/4}{\rho^2}\right]\phi(\rho) - E\phi(\rho) = 0$$

$\alpha_R = -\frac{1}{2} \quad \Longrightarrow$ Simulates **real** potentials in $d = 2$

$\langle\tau|\mathcal{P}|\tau\rangle = (-1)^k$ **oscillates**

$\beta^* = -\alpha$:   complex energies

complex "angular momentum"

like in the case of the H.O. with spontaneously broken $\mathcal{PT}$ symmetry

Further solvable periodic potentials:

Composition of step functions, δ functions, Lamé-type potentials...

Consider a simple step potential in some detail

following *V. Jakubský and M. Znojil, Czech. J. Phys.* **37** *(2004)*

$$K(\varphi) = V(\varphi) = \mathrm{i}Z\frac{\pi-\varphi}{|\pi-\varphi|} \qquad \varphi \in (0, 2\pi)$$

Main features in $d = 1$:

- As Z increases, the low-lying levels merge pairwise and their energies become complex conjugate at critical values of Z

  spontaneous breakdown of $\mathcal{PT}$ symmetry

- The energies of higher levels are close to those of the real square well

Novelties in the $d = 2$ case:

- The energies of the $d = 1$ angular potential act as angular momenta in the radial potential
- When combined with some radial potentials a whole sequence of levels merge at the same time, at various real energies

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2}\cot(\theta)\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} - V(r,\theta,\varphi)\psi + E\psi = 0 \ .$$

Search for separable variables:

$$\psi(r,\theta,\varphi) = r^{-1}\phi(r)\chi(\theta)\tau(\varphi)$$

$$\phi''\chi\tau + \frac{1}{r^2}(\chi'' + \cot(\theta)\chi') + \frac{1}{r^2\sin^2\theta}\phi\chi\tau'' - (V(r,\theta,\varphi) - E)\,\phi\chi\tau = 0$$

Assume that

$$\boxed{\tau'' = (K(\varphi) - k)\tau \qquad \text{as in } d = 2}$$

$$\boxed{\chi'' + \cot(\theta)\chi' = (Q(\theta) - q)\chi}$$

Then a radial equation is obtained

$$-\phi'' + \left[V_0(r) + \frac{q}{r^2}\right]\phi - E\phi = 0$$

where

$$V(r,\theta,\varphi) = V_0(r) + \tfrac{1}{r^2}\left(Q(\theta) + \tfrac{K(\varphi)-k}{\sin^2(\theta)}\right) \ .$$

A trivial solution for $\chi(\theta)$ is in terms of associated Legendre functions $P_\nu^\mu(\cos(\theta))$ if

$$Q(\theta) = \mu^2 \sin^{-2}(\theta) , \qquad q = \nu(\nu+1)$$

$\mathcal{PT}$-normalization of $\chi(\theta)$ is possible, if $\nu = n$, $\mu = m \leq n$:

$$\chi_{nm}(\theta) = \mathrm{i}^{n+m} \left[\left(n + \frac{1}{2} \right) \frac{(n-m)!}{(n+m)!} \right]^{1/2} P_n^m(\cos(\theta))$$

With this $\qquad \mathcal{PT}\ \chi(\theta) = \chi(\theta) \qquad\qquad \langle \chi | \mathcal{P} | \chi \rangle = (-1)^{n+m}$

$$V(r,\theta,\varphi) = V_0(r) + \frac{1}{r^2 \sin^2(\theta)} (K(\varphi) - k + m^2) \ .$$

is $\mathcal{PT}$-symmetric **if**

- $V_0(r)$ is **real**
- $K(\varphi)$ is **$\mathcal{PT}$-symmetric:** $\qquad K^*(\varphi + \pi) = K(\varphi)$
- k is real
- State-independence of $V(r,\theta,\varphi)$ requires $m^2 - k = c = const.$

BUT: $q = \nu(\nu+1)$ is **real** $\Longrightarrow$ E is also **real** $\Longrightarrow$ unbroken $\mathcal{PT}$ symmetry

$V_0(r)$ and $K(\varphi)$ can be chosen the same as in the $d = 2$ case

A more general solution of $(Q(\theta)-q)\chi(\theta)$ is also possible

$$\boxed{Q(\theta) = C\sin^{-2}(\theta) + \mathrm{i}D\cot(\theta)}$$

$$C = \left(n + \frac{1}{2}(\alpha_n + \beta_n + 1)\right)^2 = const. \qquad D = \frac{1}{2}(\alpha_n - \beta_n)(\alpha_n + \beta_n) = const.$$

$$q = \frac{1}{4}[(\alpha_n + \beta_n)^2 + (\alpha_n - \beta_n)^2 - 1)$$

This introduces an imaginary component in the potential:

$$V(r,\theta,\varphi) = V_0(r) + \frac{1}{r^2\sin^2(\theta)}(K(\varphi) - k + C) + \frac{1}{r^2}\mathrm{i}D\cot(\theta)$$

Apart from a factor of $\sin^{1/2}(\theta)$ the solutions are those of the Rosen–Morse I potential:

$$\chi_n(\theta) = c_n(1 + \mathrm{i}\cot\theta)^{\frac{\alpha_n}{2}+\frac{1}{4}}(1 - \mathrm{i}\cot\theta)^{\frac{\beta_n}{2}+\frac{1}{4}}P_n^{(\alpha_n,\beta_n)}(-\mathrm{i}\cot\theta)$$

With this non-Hermiticity can enter through the polar angle component too

q can be complex $\Longrightarrow$ spontaneous breakdown of $\mathcal{PT}$ symmetry becomes possible

Summary and outlook

- **Exact** solution of $\mathcal{PT}$-symmetric potential for $d = 2$ and 3
  Potential terms tailored to the structure of the kinetic term...
- **Separation** of the variables
- Choice for $K(\varphi)$: 2π periodic $\mathcal{PT}$-symmetric: *e.g. Scarf I, step, δ, Lamé...*
- Choice for $Q(\theta)$: from the d.e. of the $P_n^m(\cos(\theta))$, Rosen–Morse I
- Choice for $V_0(r)$: any solvable radial potential (Coulomb too!)
- Non-Hermiticity enters through the angular equations
- Complex energies originate from complex "angular momenta"
- The sign of the pseudo-norm oscillates here too
- **Degeneracies** can occur for $d \geq 1$ as in the Hermitian case
  Similar algebras can also occur
- The method and the results can be **generalized** in several ways
  Alternative choice for $\tau(\varphi)$, $\chi(\theta)$
  Higher dimensions?

The importance of $\theta\varphi$