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Pseudo-differential equations, generalised eigenvalue problems and the Bethe Ansatz

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OUTLINE

- $\sim$ Baby ODE/IM correspondence
- The 6V model, the BAE and PT-symmetric QM
- High-order differential equations from the Harmonic oscillator
- ABCD Bethe ansatz models and (pseudo)-differential equations.
- Conclusions

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$\sim$ Baby ODE/IM correspondence

Consider the QES model (Turbiner, Ushveridze, Bender-Dunne..)

$$H_{QES} = -\frac{d^2}{dx^2} + x^6 - \alpha x^2$$

with $\alpha = 2J+1$ or $J = 0, 1, 2, \dots$. Look for eigenfunctions of H_{QES} with zero-monodromy:

$$\psi(x) = \left[\prod_{i=1}^{N}(x - \varepsilon_i)\right] \exp\left(-x^4/4\right)$$

then $N = J+1$, plus a set of non-linear constraints on $\{\varepsilon_i\}$

$$\sum_{j \neq i} \frac{2}{\varepsilon_j - \varepsilon_i} + \varepsilon_i - \frac{3}{2\varepsilon_i} = 0$$

and $E_n \equiv E_n(\{\varepsilon_i\}) \quad n = 1, \dots N$. This set of constraints coincides with the Bethe ansatz equations for the (generalised) Gaudin model!! (An finite N-site quantum spin chain). ODE/IM is a correspondence between

Quantum Integrable Hamiltonians

and

ODEs+zero-monodromy conditions

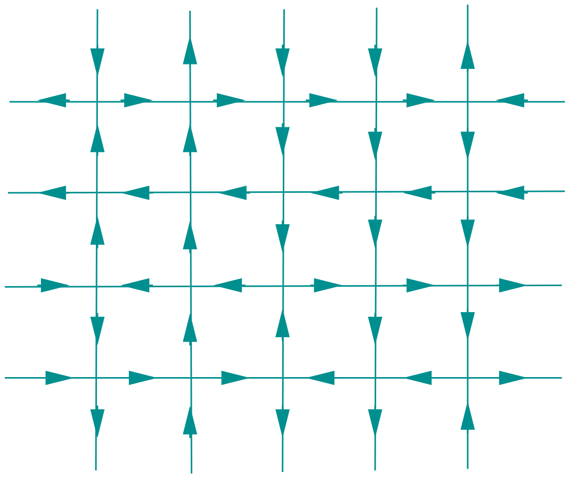
(On the ODEs, except $x = 0$ and $x = \infty$). It is the quantum version of

Gaudin models $\leftrightarrow$ Classical zero-monodromy Opers

(Feigin and Frenkel [arXiv:0705.2486]).

The 6V model the BAE and, PT-symmetric QM

Consider $N \times M$ lattice model with periodic BCs and $N/2$ even. On each link of the lattice, we place a spin



A) Only those configurations of spins which preserve the 'flux' of arrows through each vertex are allowed.

B) We shall only consider the zero field 6-V model which has an additional '4-spin reversal' symmetry. Locally this gives six options:

$$W\begin{bmatrix} & \uparrow & \\ \rightarrow & & \rightarrow \\ & \uparrow & \end{bmatrix} = W\begin{bmatrix} & \downarrow & \\ \leftarrow & & \leftarrow \\ & \downarrow & \end{bmatrix} = a$$

$$W\begin{bmatrix} & \downarrow & \\ \rightarrow & & \rightarrow \\ & \downarrow & \end{bmatrix} = W\begin{bmatrix} & \uparrow & \\ \leftarrow & & \leftarrow \\ & \uparrow & \end{bmatrix} = b$$

$$W\begin{bmatrix} & \uparrow & \\ \rightarrow & & \leftarrow \\ & \downarrow & \end{bmatrix} = W\begin{bmatrix} & \downarrow & \\ \leftarrow & & \rightarrow \\ & \uparrow & \end{bmatrix} = c$$

The overall normalisation factors out trivially from all calculations, and we can parametrise the remaining two degrees of freedom using:

ν (the spectral parameter)

η (the anisotropy)

as

$$a = \sinh(i\eta - \nu)\ , \quad b = \sinh(i\eta + \nu)\ , \quad c = \sinh(2i\eta)$$

To calculate the partition function Z , define the transfer matrix , $\mathbf{T}$:

$$\mathbf{T}^{\{\alpha'\}}_{\{\alpha\}}(\nu) \;=\; \sum_{\{\beta_i\}} W\left[\beta_1 \, {}^{\alpha'_1}_{\alpha_1} \, \beta_2\right] W\left[\beta_2 \, {}^{\alpha'_2}_{\alpha_2} \, \beta_3\right] \ldots W\left[\beta_N \, {}^{\alpha'_N}_{\alpha_N} \, \beta_1\right]$$

In terms of $\mathbf{T}$ the partition function is

$$Z = \text{Trace}\left[\mathbf{T}^M\right] \ .$$

The free energy per site in the limit $M \to \infty$ can be obtained as

$$f = \frac{1}{NM} \ln Z = \frac{1}{NM} \ln \text{Trace}\left[\mathbf{T}^M\right] \sim \frac{1}{N} \ln t_0 \ ,$$

where $t_0 \equiv t$ is the ground-state eigenvalue of $\mathbf{T}$.
Baxter's T-Q relation : there exists an auxiliary function $q(\nu)$

$$q(\nu) = \prod_{n=0}^{N/2-1} \sinh(\nu - \nu_l) \ ,$$

such that

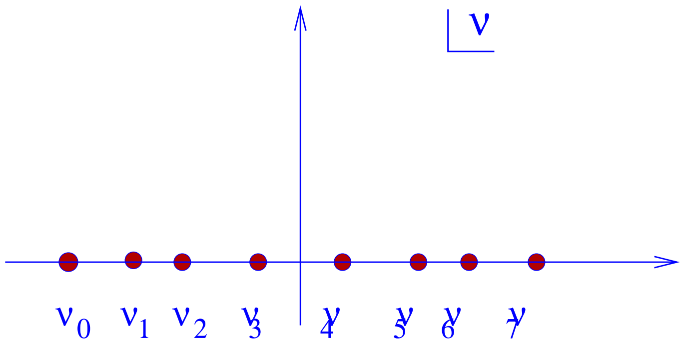
$$t(\nu)q(\nu) = a(\nu,\eta)^N q(\nu + 2i\eta) + b(\nu,\eta)^N q(\nu - 2i\eta)$$

BAE then emerge as a *consequence* of the fact that both t_0 and q are entire. Setting

$$q(\nu_i) = 0 \ ,$$

we find

$$-1 = \frac{a^N(\nu_i,\eta)}{b^N(\nu_i,\eta)} \frac{q(\nu_i + 2i\eta)}{q(\nu_i - 2i\eta)} \ .$$



The conformal limit is achieved by sending

$$N \to \infty \quad \text{and} \quad a = e^{\pi\nu/2\eta} \to 0 \ ,$$

with aN finite. Defining

$$\lambda_i = e^{2\nu_i} \quad , \quad \Omega = e^{i4\eta} \ ,$$

the λ_i 's for $i \ll \ln N$ rescale to zero as

$$\lambda_i \sim E_i a^{4\eta/\pi} \sim E_i N^{-4\eta/\pi} \ ,$$

and the BAE becomes for $\pi/4 < \eta < \pi/2$

$$-1 = \prod_{n=0}^{\infty} \frac{(E_n - E_i\Omega)}{(E_n - E_i\Omega^{-1})} \ .$$

Twisted BCs:

$$-1 \Longrightarrow -e^{2i\phi}$$

and the T-Q relation becomes

$$t(E,\phi)q(E,\phi) = e^{i\phi}q(\omega^2 E,\phi) + e^{-i\phi}q(\omega^{-2}E,\phi) \ ,$$

ODE/IM result:

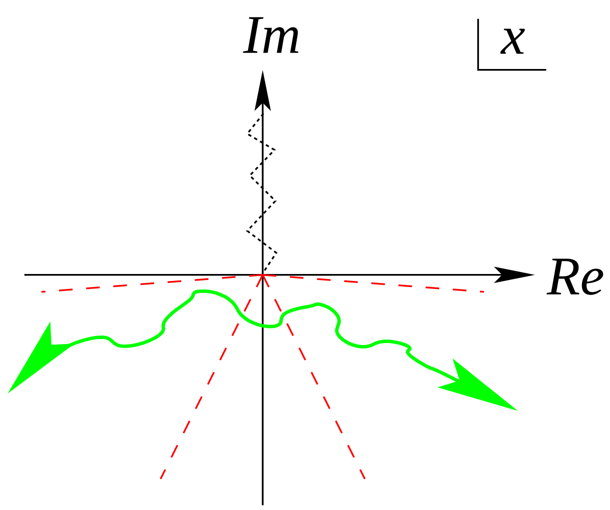
$$t(-E,\phi) \leftrightarrow \text{Spect. det. PT-symmetric QM}$$

for

$$\mathcal{H}_{M,l} = p^2 - (ix)^{2M} + l(l{+}1)/x^2$$

(M and l real, $M > 0$.)

This amounts to studying the effect of an angular-momentum-like term $l(l{+}1)x^{-2}$ on the Bender-Boettcher problem.



Integrable Model		Schrödinger equation
ν	$\leftrightarrow$	Energy
η	$\leftrightarrow$	Mπ /(2M+2)
ϕ	$\leftrightarrow$	(2l+1)π /(2M+2)
t	$\leftrightarrow$	Lateral spectral problems defined at $\lvert x\rvert=\infty$
q	$\leftrightarrow$	Radial spectral problems linking $\lvert x\rvert=\infty$ and $\lvert x\rvert=0$

At $M = 1$

$$t(-E, l) = \frac{2\pi}{\Gamma(\frac{1}{2} + \frac{2l+1-E}{4})\Gamma(\frac{1}{2} - \frac{2l+1+E}{4})}$$

and the ‘PT-Harmonic oscillator’ spectrum is real

$$E \in \{3 + 2l + 4n\} \cup \{1 - 2l + 4n\} \quad (n = 0, 1, \ldots) \ .$$

Consider

$$\left[-\frac{d^2}{dx^2} + \frac{1}{4}(x^2 - \lambda)\right] \psi(x) = 0$$

impose vanishing BCs at $x \to +\infty$

$$\psi(x)_{x\to\infty} \sim x^{-\frac{1}{2}+\lambda} e^{-\frac{1}{2}x^2}$$

and set

$$\Psi(x, \lambda, \lambda') = \psi(x, \lambda)\psi(x, \lambda')$$

Two possible behaviors as $x \to -\infty$. In general

$$\psi(x,\lambda)_{x\to-\infty} \sim (-x)^{-\frac{1}{2}-\lambda} \exp(+\frac{1}{2}x^2)$$

exceptionally at $\lambda_i = \lambda \in \{3+4n\} \cup \{1+4n\}$ $(n = 0, 1, \ldots)$

$$\psi(x,\lambda)_{x\to-\infty} \sim (-x)^{-\frac{1}{2}+\lambda_i} \exp(-\frac{1}{2}x^2)$$

therefore

$$\Psi(x,\lambda_i,\lambda') = \psi(x,\lambda_i)\psi(x,\lambda')$$

can be (ϵ-regularised) Fourier transformed

$$\tilde{\Psi}(k) = \mathcal{F}[\Psi(x)] = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} d\theta \, \Psi(x) e^{-ikx+\epsilon x} \; .$$

Let's see what kind of ODE $\Psi(x,\lambda,\lambda')$ satisfies:

$$\Psi''' = (x^2 - \bar{\lambda})\Psi' + x\Psi - \frac{(\Delta\lambda)^2}{4}\left(\frac{d}{dx}\right)^{-1}\Psi$$

where

$$\bar{\lambda} = (\lambda + \lambda')/2 \; , \qquad \Delta\lambda = (\lambda - \lambda')/2$$

where the pseudo-differential operator (introduced for later convenience) is defined by its formal action

$$\left(\frac{d}{dx}\right)^{-1} x^s = \frac{x^{s+1}}{s+1}$$

Fourier transforming the final equation and, after gauge transformation

$$-\frac{d^2}{dp^2} + (p^2 - \bar{\lambda}) + \frac{(\Delta\lambda)^2 - 1}{4x^2}$$

this is our initial PT-symmetric HO with

$$E = (\lambda + \lambda')/2$$

$l = 1/2(-1 - \lambda/2 + \lambda'/2)$ or $l = 1/2(-1 - \lambda'/2 + \lambda/2)$.

The PT-spectral determinant is

$$t(-E, \lambda, \lambda') = \frac{2\pi}{\Gamma(\frac{1}{2} - \frac{\lambda}{4})\Gamma(\frac{1}{2} - \frac{\lambda'}{4})} \ .$$

Starting from

$$\left[-\frac{d^2}{x^2} + \frac{1}{4}(x^2 - \lambda + \frac{k}{x^2}) \right] \psi(x, \lambda, k) = 0$$

and setting

$$\Psi(x, \lambda, \rho, \sigma) = \psi(x, \lambda, \rho)\psi(x, \lambda, \sigma)$$

the resulting pseudo-differential operator has a term

$$-\frac{(\rho - \sigma)^2}{16x^2}(\frac{d}{dx})^{-1}\frac{1}{x^2} \ .$$

This case is related to a $SO(4)$ particular in a $SO(2n)$ family of high-order ordinary pseudo-differential equations (Dorey, Dunning, Masoero, Suzuki, Tateo (2006)).

What we have learned?

- The seemingly unphysical initial problem has a nice interpretation in terms of a pair of standard Harmonic oscillators

  ($\sim$ Pais-Uhlenbeck oscillator model studied by Bender and Mannheim in arXiv:0706.0207 [hep-th]).

- Certain high-order ODEs with real spectra should be considered more seriously!

ABCD Bethe ansatz models and (pseudo) ODEs

For the A to G simple Lie algebras, the general CFT Bethe ansatz equations are

$$\prod_{b=1}^{r} \Omega^{C_{ab}\gamma_b} \frac{Q^{(b)}(\Omega^{C_{ab}}E_i^{(b)},\gamma)}{Q^{(b)}(\Omega^{-C_{ab}}E_i^{(b)},\gamma)} = -1\,, \qquad i = 1,2,\ldots$$

where $r = \mathrm{rank}(g)$ $(g \in \{A_n,\ldots,G_2\})$, C_{ab} is the matrix

$$C_{ab} = \frac{\langle a|b\rangle}{\langle max|max\rangle}\ ,$$

$|a\rangle$ and $|b\rangle$ are simple roots and

$$\langle max|max\rangle = \mathrm{Max}(\langle j|j\rangle) \quad (j = 1,\ldots,r)\ .$$

We parametrise Ω in terms of a real number $\mu > 0$ as

$$\Omega = \exp\left(i\frac{2\pi}{h^\vee \mu}\right)\ .$$

The roots of the BAE split into 'multiplets' with equal $|E_i^{(a)}|$ (strings).

The ground-state of the original quantum spin chain usually corresponds to *pure* configurations with 'multiplets' of the same dimension d (d -strings).

In the spin-j $su(2)$ quantum chains $d = 2j$.

We first introduce the n^{th} -order differential operator

$$D_n(\mathbf{g}) = D(g_{n-1} - (n-1)) \ldots D(g_1 - 1)\, D(g_0) \ ,$$

$$D(g) = \left(\frac{d}{dx} - \frac{g}{x}\right) \ ,$$

$\mathbf{g} = \{g_{n-1}, \ldots, g_1, g_0\}$, $\mathbf{g}^\dagger = \{n-1-g_0, \ldots, n-1-g_{n-1}\}$, and

$$P_K(E,x) = (x^{h^\vee M/K} - E)^K \ .$$

PS: $K > 1$ in A_1 , $d = K = 2j$: Lukyanov's idea!

In general

$$d = K/C_{11} \ .$$

The relevant pseudo-differential equations are:

su(n):

$$((-1)^n D_n(\mathbf{g}) - P_K)\psi(x) = 0 \;;$$

so(2n):

$$\left(D_n(\mathbf{g}^\dagger) \left(\frac{d}{dx} \right)^{-1} D_n(\mathbf{g}) - \sqrt{P_K} \left(\frac{d}{dx} \right) \sqrt{P_K} \right) \psi(x) = 0 \;;$$

so(2n+1):

$$\left(D_n(\mathbf{g}^\dagger) D_n(\mathbf{g}) + \sqrt{P_K} \left(\frac{d}{dx} \right) \sqrt{P_K} \right) \psi(x) = 0 \;;$$

sp(2n):

$$\left(D_n(\mathbf{g}^\dagger) \left(\frac{d}{dx} \right) D_n(\mathbf{g}) - P_K \left(\frac{d}{dx} \right)^{-1} P_K \right) \psi(x) = 0 \;;$$

$$A_3 \Longleftrightarrow D_3$$

$$D_2 \Longleftrightarrow A_1 \oplus A_1$$

$$B_1 \Longleftrightarrow A_1 \qquad \text{(K even)}$$

$$(\ g_i = i,\ M = \tfrac{2}{h^\vee},\ K = 1) \quad D_n, B_n \Longleftrightarrow A_1$$

$$B_2 \Longleftrightarrow C_2$$

Dualities

$$A_{-n} \leftrightarrow A_n \qquad (K \leftrightarrow -K)$$

and

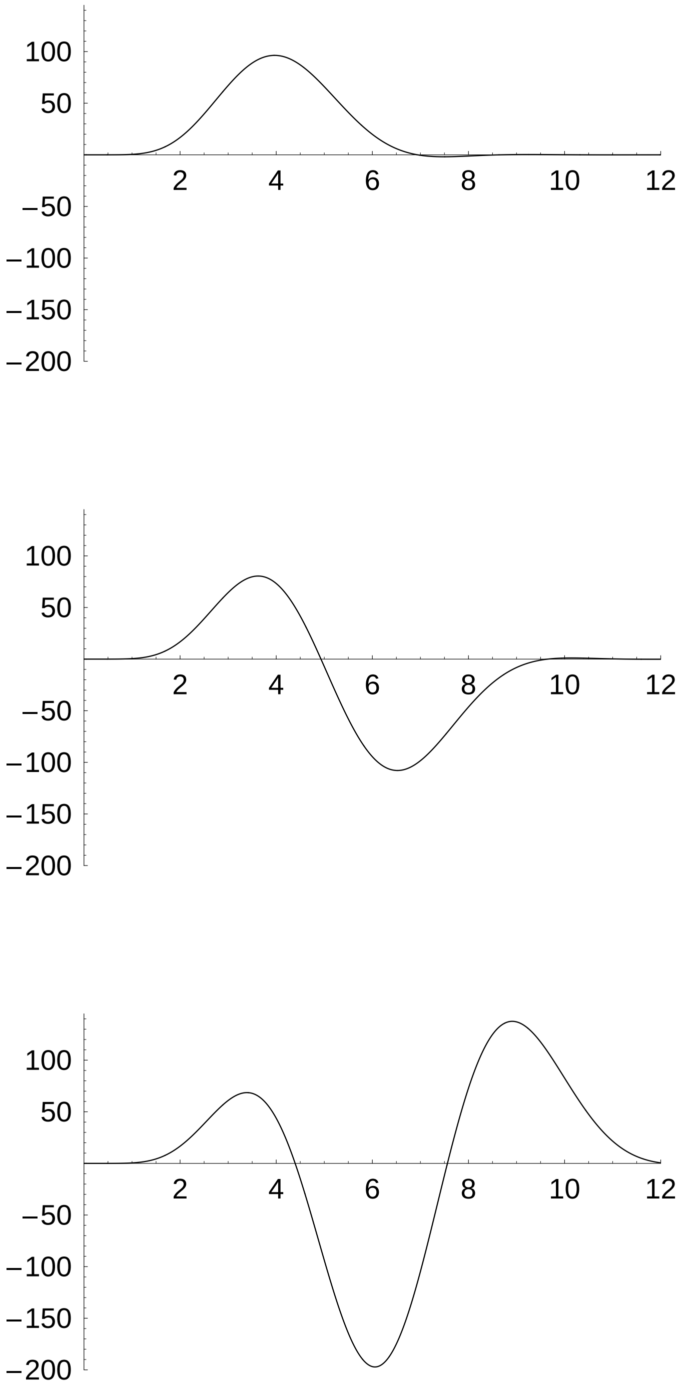
$$D_{-n} \leftrightarrow C_n \qquad (K \leftrightarrow -K/2)$$

similar to W-algebra dualities (Hornfeck 1994)

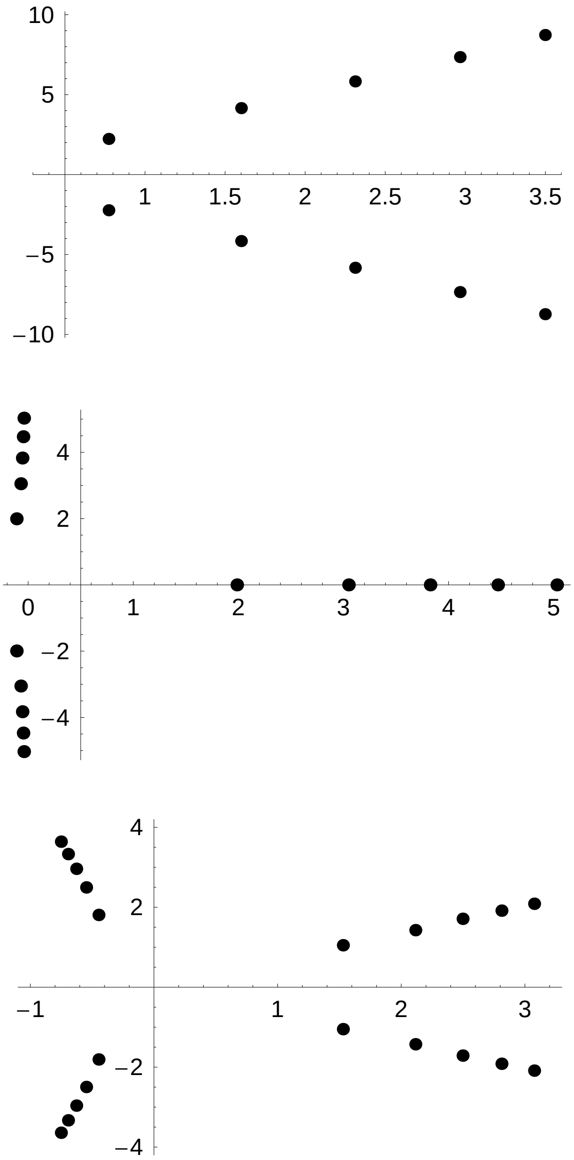
$$\frac{\hat{su}(-n)_K \times \hat{su}(-n)_\mu}{\hat{su}(-n)_{K+\mu}} \leftrightarrow \frac{\hat{su}(n)_{-K} \times \hat{su}(n)_{\bar{\mu}}}{\hat{su}(n)_{-K+\bar{\mu}}}$$

$$\frac{\hat{so}(-2n)_K \times \hat{so}(-2n)_\mu}{\hat{so}(-2n)_{K+\mu}} \leftrightarrow \frac{\hat{sp}(2n)_{-K/2} \times \hat{sp}(2n)_{\bar{\mu}}}{\hat{sp}(2n)_{-K/2+\bar{\mu}}}$$

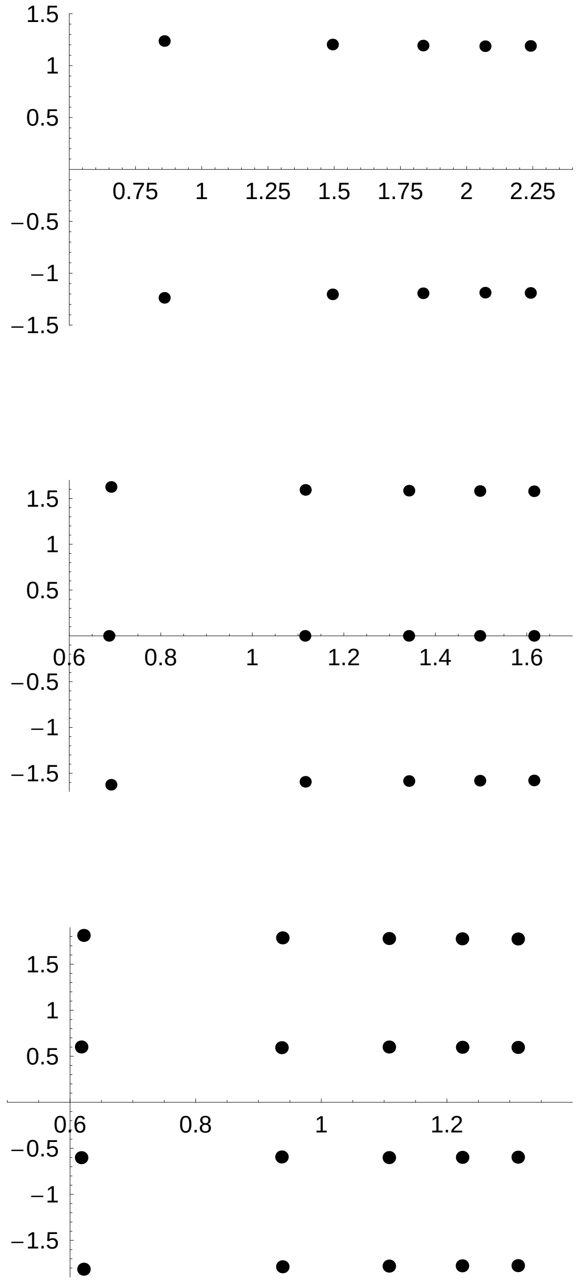
(see also Cvitanovic E-book)



Lowest three functions $\Psi(x, E)$ for a D_4 pseudo-differential equation.



Complex E-plane: the eigenvalues for the $SU(2)$ model with $M = 3$, $g_0 = 0$ for $K = 2$, 3 and 4 respectively.



Complex (ln E)-plane: two, three- and four-strings.

Conclusions

Maths: connection with classical W-algebras, Opers in generalised KdV equations.

Physics: PT-symmetric QM, applications to condensed-matter physics.